Exploring Graph State Local Equivalence Classes with Distance Hereditary Split Decompositions

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Abstract

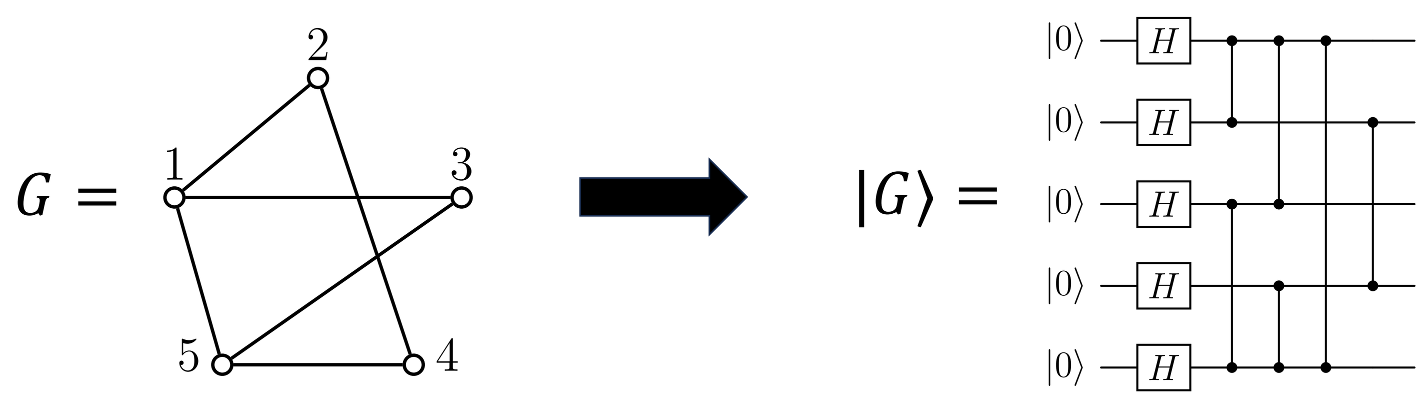
Graph states are an important resource for measurement-based quantum computation[1]. Single-qubit Clifford operations change the state while preserving the entanglement structure, a process modeled by *local complement* (LC) operations on graphs[2]. This research explores the LC equivalence class for certain graphs.

Recent research into graph states has focused on enumerating LC equivalent graphs[3], often with a goal such as minimizing the ammount of preparation resources[4]. While previous approaches have relied on brute force searching, we use a technique known as the *split decomposition*[5] to characterize the entire equivalence class for certain families of *distance hereditary* graphs[6].

Graphs and Graph States

The preparation of a *graph state* is represented by a *graph*[1].

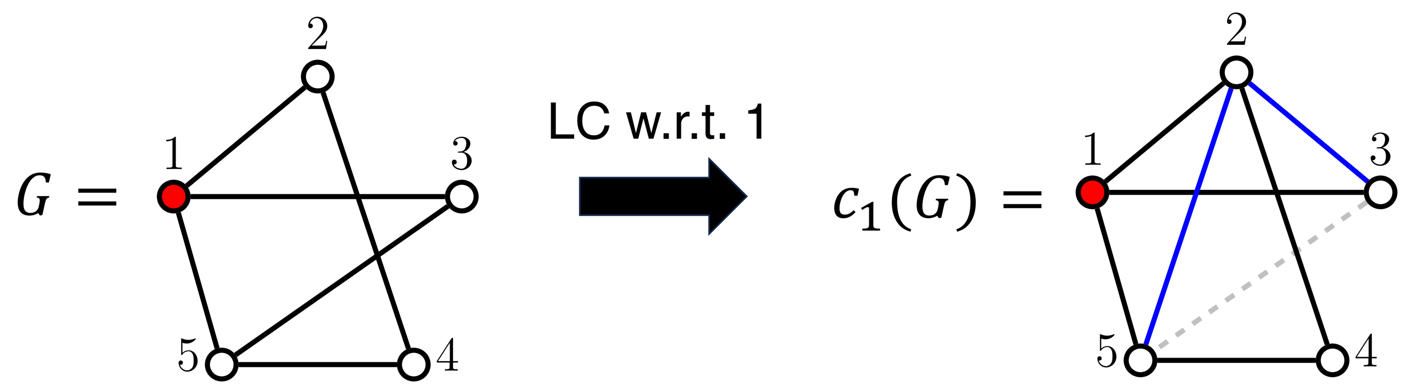
- Vertices denote qubits.
- Edges denote entanglement through a CZ gate.



Graphs with fewer edges require fewer CZ gates to implement.
Graphs with lower vertex degrees require fewer time steps[4].

Graph Local Equivalence

Local complement (LC) is a transformation on graphs[7].
LC with respect to a vertex v complements the edge set of $N(v)$.

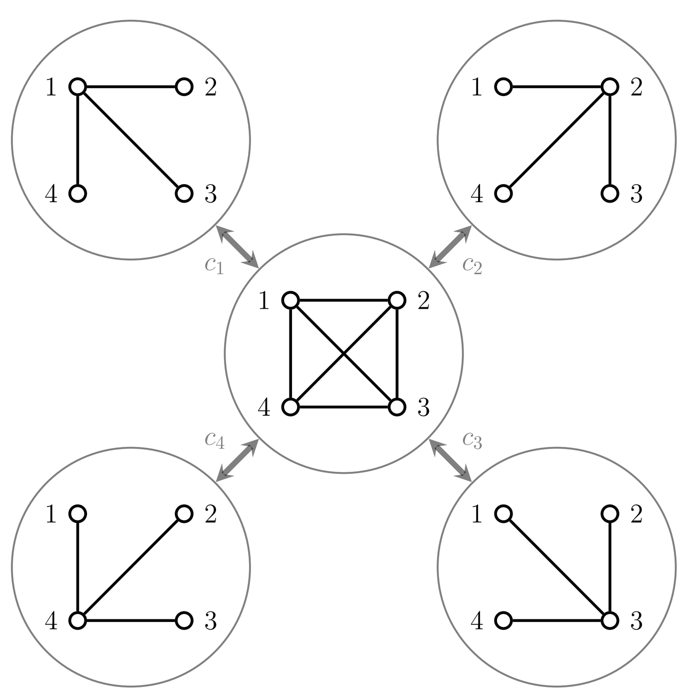


Graphs related via LC operations are called *locally equivalent*.
Corresponding graph states are related via single-qubit Cliffords.

Local Complement (LC) Orbits

The LC orbit of a graph refers to the *equivalence class* of graphs related by LC operations.
We examine symmetries to derive explicit formulas for the orbit size.

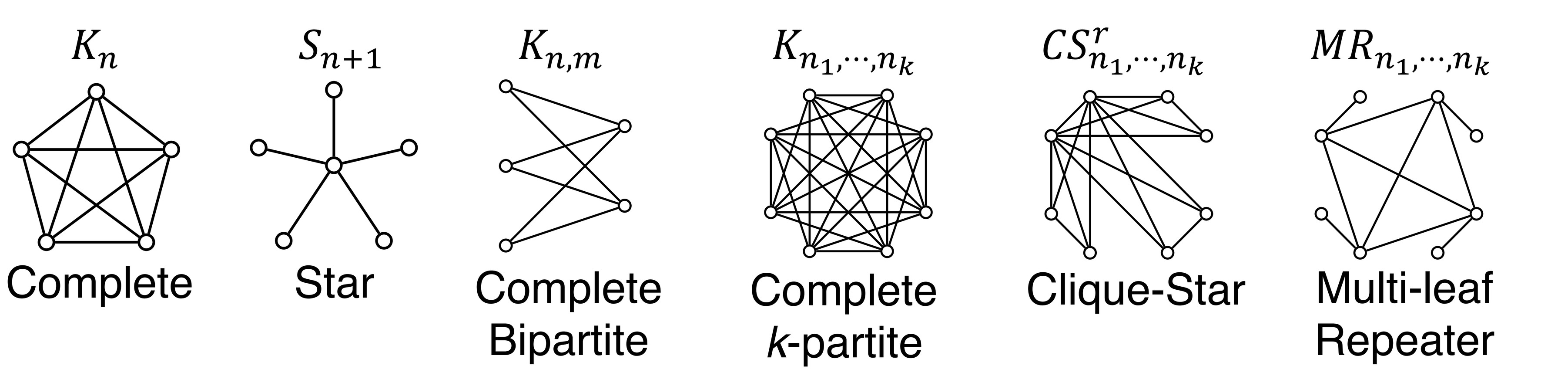
Example: Complete Graph K_n
LC orbit contains $n + 1$ graphs.



Distance Hereditary (DH) Graphs

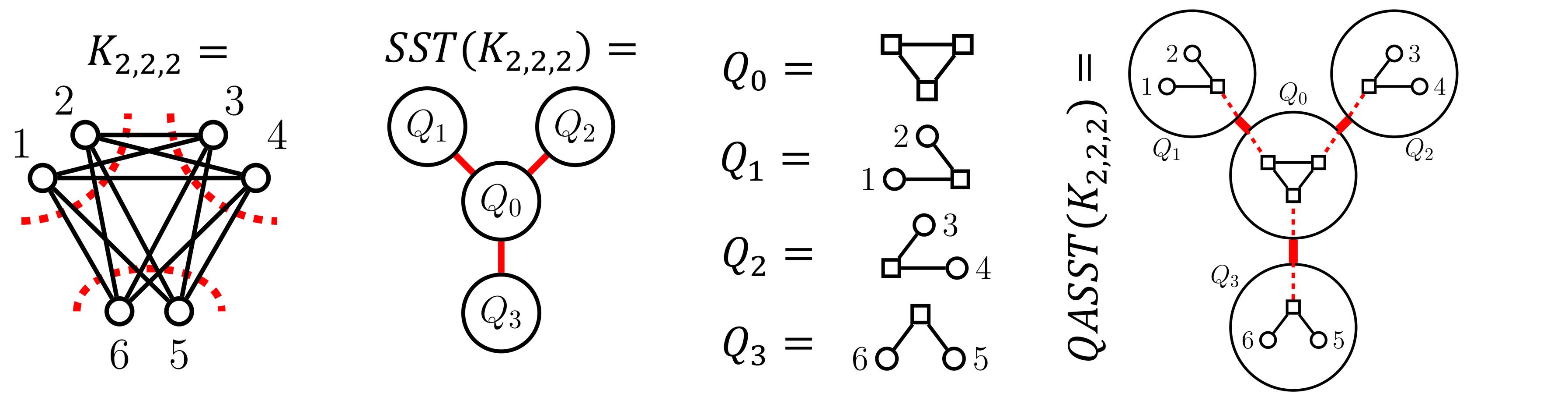
A graph is called *distance hereditary* (DH) if all connected induced subgraphs preserve distance between vertices[6].
LC operations preserve the distance hereditary property.

Equivalent: completely separable; recursive construction via twins; rank-width 1



Split Decompositions of DH Graphs

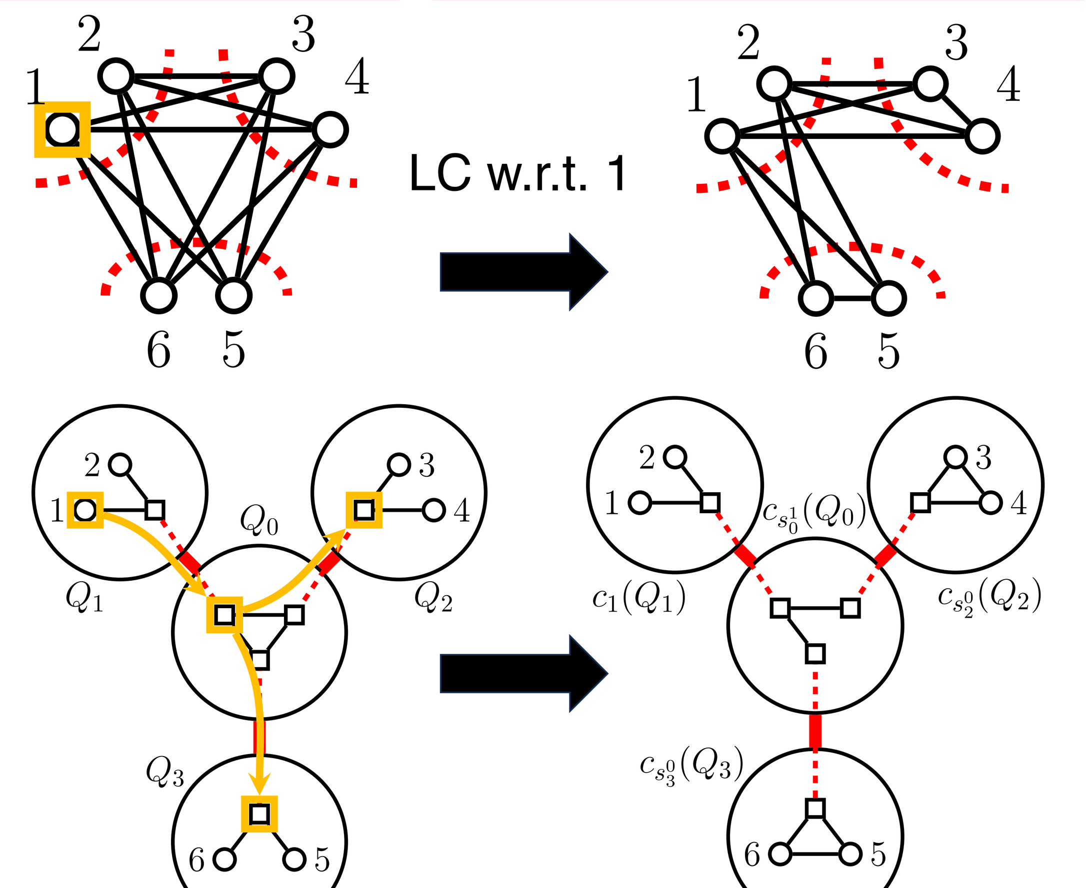
The *split decomposition* divides a graph into simpler components by collapsing edges between *complete bipartite subgraphs*[5]. The connectivity of these components, called *quotient graphs*, is described via the *quotient-augmented strong split tree* (QASST).



Quotient graphs have three types: *star*, *complete*, or *prime*.
Distance hereditary graphs contain only *star* or *complete.*
Local complement preserves the tree and transforms quotients.

LC Propagation through QASST

LC operations propagate through quotient graphs via adjacent *split-nodes*. This process is recursive.



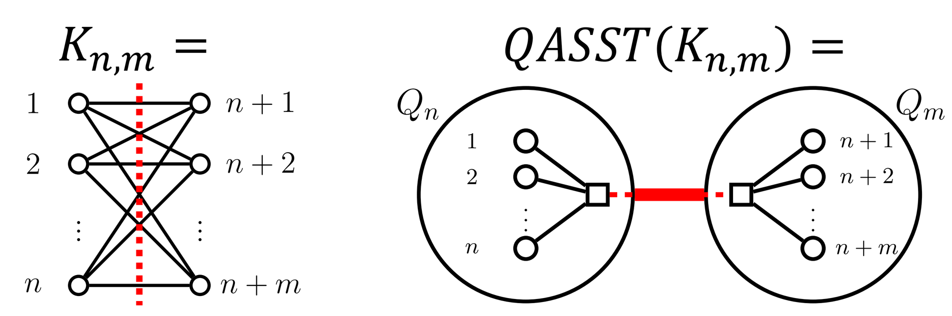
Algorithm 1: LC Propagation

Data: A vertex v, quotient graphs $Q_1, \cdots, Q_k$.
Result: Quotient graphs $Q'_1, \cdots, Q'_k$.

for $\ell = 1, \cdots, k$ **do**
  Initialize $Q'_\ell = Q_\ell$;
Identify Q_i for which $v \in V(Q_i)$;
Update $Q'_i = c_v(Q_i)$;
for *split-node* $s_i^j \in N_{Q_i}(v)$ **do**
  Update $(Q'_1, \cdots, Q'_k) =$ **LC Propagation**$(s_j^i, Q'_1, \cdots, Q'_k)$;
Return $Q'_1, \cdots, Q'_k$;

Complete Bipartite QASST Symmetries

Enumeration of all quotient graph symmetris in QASST of complete bipartite graph.



Q_n	(count)	Q_m	(count)	Total	Transformation from $K_{n,m}$	Number of Edges
c	1	c	1	1	invalid	N/A
sc	1	ss	m	m	invalid	N/A
ss	n	sc	1	n	invalid	N/A
sc	1	sc	1	1	$\mathrm{id}(K_{n,m})$	nm
sc	1	c	1	1	$c_{i^n}(K_{n,m})$	$nm + \frac{m(m-1)}{2}$
c	1	sc	1	1	$c_{i^m}(K_{n,m})$	$nm + \frac{n(n-1)}{2}$
ss	n	c	1	n	$c_{i^n} \circ c_{i^m}(K_{n,m})$	$n + m - 1 + \frac{m(m-1)}{2}$
c	1	ss	m	m	$c_{i^m} \circ c_{i^n}(K_{n,m})$	$n + m - 1 + \frac{n(n-1)}{2}$
ss	n	ss	m	nm	$c_{i^n} \circ c_{i^m} \circ c_{i^n}(K_{n,m})$	$n + m - 1$

orbit size
min. edges

LC Orbit Sizes for Selected DH Graphs

Class of Graph	Notation	Size of the LC Orbit
Complete Bipartite	$K_{n,m}$	$nm + n + m + 3$
Complete *k*-partite	$K_{n_1,\cdots,n_k}$	$\Pi_{i=1}^k(n_i+1) + \Sigma_{j=1}^k \Pi_{i\in[k]\setminus j}(n_i+1)$
Clique-Star	$CS^r_{n_1,\cdots,n_k}$	$\Pi_{i=1}^k(n_i+1) + \Sigma_{j=1}^k \Pi_{i\in[k]\setminus j}(n_i+1)$
Multi-leaf Repeater	$MR_{n_1,\cdots,n_k}$	$K_{n_1,\ldots,n_k}$ (even *k*) or $CS^r_{n_1,\ldots,n_k}$ (odd *k*)

**even* products only; **odd* products only; two possibilities for multi-leaf repeater.

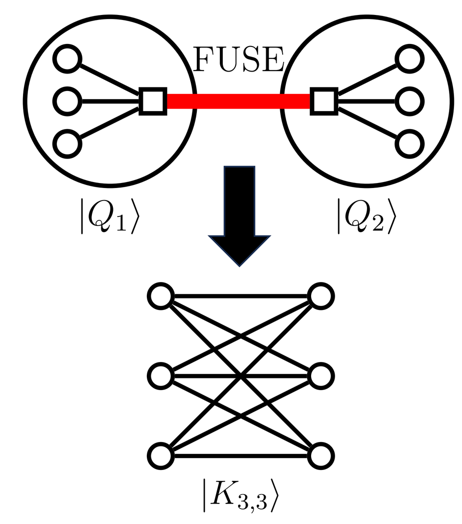
Efficient Graph State Preparation Method

Graph state creation via *fusion* on QASST:

- Implement quotients as graph states $|Q_1\rangle, \cdots, |Q_k\rangle$.
- Fuse split-node qubits into full graph state $|G\rangle$.

For a DH graph with n vertices and k quotients:

- Prepare quotient graph states as *stars* (minimal edge).
- Requires $n + 2k - 2$ qubits (including split-nodes).
- Apply local Cliffords to quotients (at most k), then fuse.
- Requires $n + 2k - 3$ total CZ gates (including fusions).



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Acknowledgements

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