Describing Non-Algebraic Tangles with Graphs

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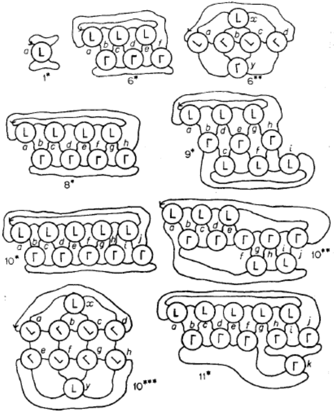
Connection to Knot Theory: Conway's Tabulation[1]

- Conway introduced tangles as a way to tabulate knots.
- Tangles are regarded as portions of knot diagrams.
    - planar polyhedra describe diagrams
    - vertices represent tangles
- All knots up to 11 crossings can be described this way.
    - basic polyhedra (right)
    - algebraic tangle vertices
- Improved on by Caudron.



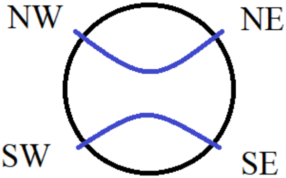
[1] J. H. Conway. An enumeration of knots and lints, and some of their algebraic properties. In *Computational Problems in Abstract Algebra (Proc. Conf., Oxford 1967)*, pages 329-358. Pergamon, Oxford, 1970

Definition

Definition

An **n-string tangle** consists of a 3-dimensional ball with a fixed boundary which has *n* strings properly embedded inside.

- Endpoints are fixed on the surface of the ball.
- For 2-string tangles, label these NW, NE, SE, SW.



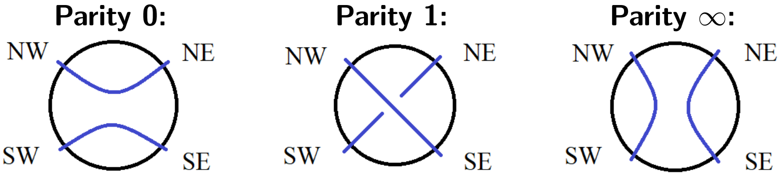
Tangle Diagrams and Parity

Definition

A **tangle diagram** is a 2-dimensional projection of a tangle.

- Crossings distinguish over-strand and under-strand.
- Endpoints are identified by NW, NE, SE, SW (2-string).

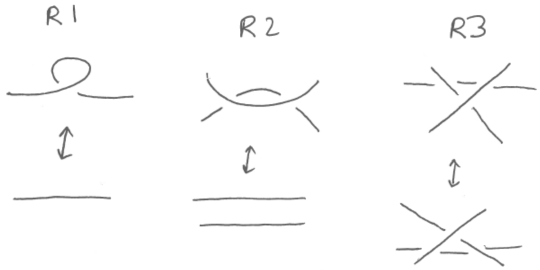
A 2-string tangle has three possible **parities**.



Equivalent Tangles and Reidemeister Moves

Definition

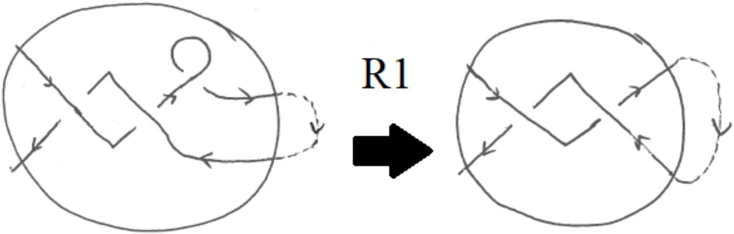
Two tangles are said to be **equivalent** if there exists an ambient isotopy between them which leaves the boundary of the ball fixed.



Two tangle diagrams are equivalent if and only if they are related by a sequence of **Reidemeister moves**.

Equivalent Tangles and Reidemeister Moves

Example:



These are equivalent tangle diagrams related by an R1 move.

Tangle Closure

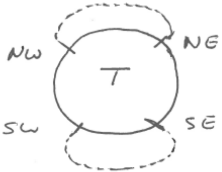
Definition

The **closure** of a 2-string tangle joins the endpoints:

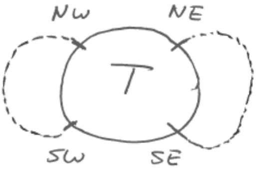
- **numerator:** join NW-NE and SW-SE.
- **denominator:** join NW-SW and NE-SE.

The closure forms a knot or link.

Numerator Closure:



Denominator Closure:



Tangle Orientation

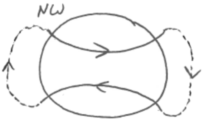
Definition

An **orientation** on a tangle is a choice of direction for each string.

Orientation convention for 2-string tangles:

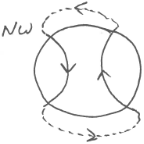
- Travel inward from NW endpoint.
- Use denominator closure for parity 0 or 1.
- Use numerator closure for parity ∞.

Parity 0:



Parity 1:

Parity ∞:



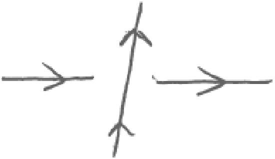
Crossing Sign

Definition

The **sign** of an oriented crossing is determined by the direction of the overstrand and understrand.

Positive Crossing:

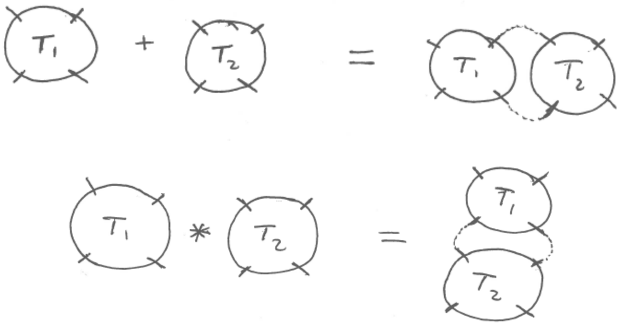
Negative Crossing:



Sum and Product of 2-String Tangles

Definition

A **sum** of tangles $T_1 + T_2$ joins the tangles horizontally as shown.
A **product** of tangles $T_1 * T_2$ joins the tangles vertically as shown.



Tangle Diagram Notations

Dowker Code:

$$-4 \mid -6 \quad -2$$

Gauss Code:

$$|)a1+b2+(|)a3+b1+a2+b3+(|$$

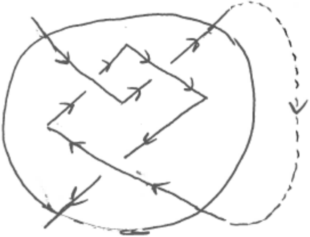
Planar Diagram Code:

$$\begin{array}{ccccc} + & b2 & a3 & d3 & c2 \\ + & b3 & a1 & d1 & c3 \\ + & b1 & a2 & d2 & c1 \end{array}$$

Coloring Matrix:

$$\begin{bmatrix} 1 & -2 & 0 & 1 & 0 \\ -2 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & -2 & 1 \end{bmatrix}$$

Example:



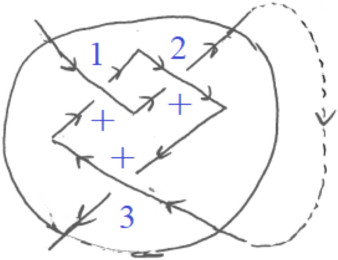
Gauss Code: Algorithm

Algorithm:

1. Start at NW corner.
2. Index crossings with subsequent integers.
3. For each crossing **encounter**, record:
   - above/below (a or b)
   - crossing index
   - crossing sign
4. List crossing information.
5. Place bar between strings.

Example:

$$|)\ \underbrace{\overbrace{a1+}^{\text{crossing}}\ b2+}_{\text{string 1}}(|)\ \underbrace{a3+b1+a2+b3+}_{\text{string 2}}(|$$

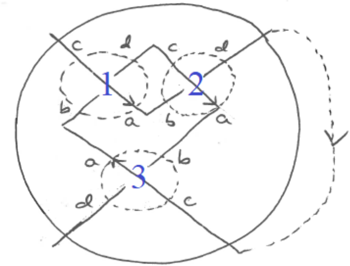


Planar Diagram Code

Algorithm:

Example:

$+$	$b2$	$a3$	$d3$	$c2$
$+$	$b3$	$a1$	$d1$	$c3$
$+$	$b1$	$a2$	$d2$	$c1$



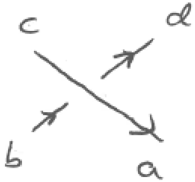
Planar Diagram Code: Labeling Convention

Crossing Labeling Convention:

- Each crossing has four corners.
- Label outward pointing overstrand a.
- Label remaining corners clockwise by b, c, d.

Corner Connections:

Example:



Planar Diagram Code: Labeling Convention

Crossing Labeling Convention:

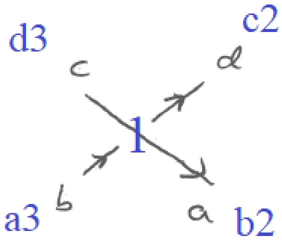
- Each crossing has four corners.
- Label outward pointing overstrand *a*.
- Label remaining corners clockwise by *b*, *c*, *d*.

Corner Connections:

- Index each crossing.
- Record crossing sign.
- For each corner, record connected crossing.

$$\underbrace{+}_{\text{sign}}\ \underbrace{b2}_{a}\ \underbrace{a3}_{b}\ \underbrace{d3}_{c}\ \underbrace{c2}_{d}$$

Example:



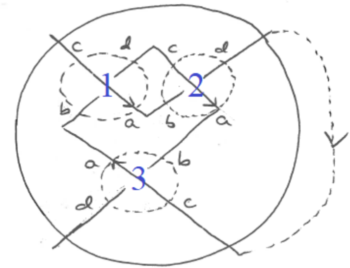
Planar Diagram Code: Algorithm

Algorithm:

1. Start at NW corner.
2. Index crossings with subsequent integers.
3. Label corners of each crossing a, b, c, d.
4. Record corner connections for each crossing.
5. List rows with corner connection information.

Example:

1	$+$	$b2$	$a3$	$d3$	$c2$
2	$+$	$b3$	$a1$	$d1$	$c3$
3	$+$	$b1$	$a2$	$d2$	$c1$



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Hierarchy of 2-String Tangle Types

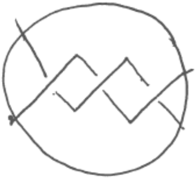
$$
\begin{array}{c}
\{\text{arbitrary 2-string tangles}\} \\
\cup\!| \\
\{\text{algebraic tangles}\} \\
\cup\!| \\
\{\text{generalized Montesinos tangles}\} \\
\cup\!| \\
\{\text{Montesinos tangles}\} \\
\cup\!| \\
\{\text{rational tangles}\} \\
\cup\!| \\
\{\text{integer tangles}\}
\end{array}
$$

Integer Tangles

Definition

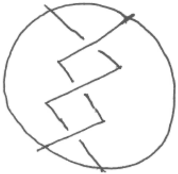
An **integer tangle** is constructed from any sequence of horizontal or vertical twists.

Example: $3/1$



horizontal

Example: $1/3$



vertical

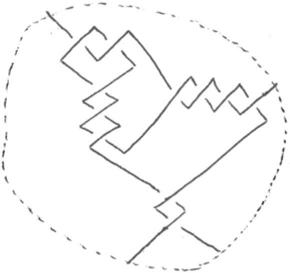
Rational Tangles

Definition

A **rational tangle** is constructed from an alternating sequence of horizontal and vertical twists, denoted by a **twist vector**.

Example: rational tangle with canonical* diagram

- 2 horizontal twists
- 3 vertical twists
- 4 horizontal twists
- 2 vertical twists
- 0 horizontal twists



Twist Vector:

$$(2, 3, 4, 2, 0)$$

Rational Tangles: Bijection with Extended Rationals

Theorem

Rational tangles (up to equivalence with canonical form) are in bijection with the extended rational numbers.[2]

- Every rational tangle can be placed in canonical form.
- Every canonical form can be described with a twist vector.
- Each twist vector defines a continued fraction in $\mathbb{Q} \cup \{\infty\}$.

$$(a_1, a_2, \cdots, a_{n-1}, a_n) \qquad \leftrightarrow \qquad a_n + \cfrac{1}{a_{n-1} + \cdots + \frac{1}{a_2 + \frac{1}{a_1}}}$$

- This fraction is a rational tangle invariant.

[2] J. H. Conway. An enumeration of knots and lints, and some of their algebraic properties. In *Computational Problems in Abstract Algebra (Proc. Conf., Oxford 1967)*, pages 329-358. Pergamon, Oxford, 1970

Rational Tangles: Example

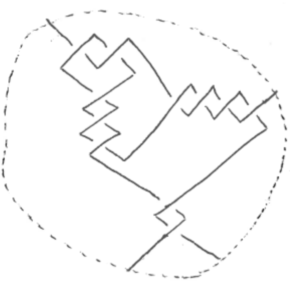
Twist Vector:

$$(2,3,4,2,0)$$

Fraction:

$$0+\cfrac{1}{2+\cfrac{1}{4+\cfrac{1}{3+\frac{1}{2}}}} = \frac{30}{67}$$

Diagram:

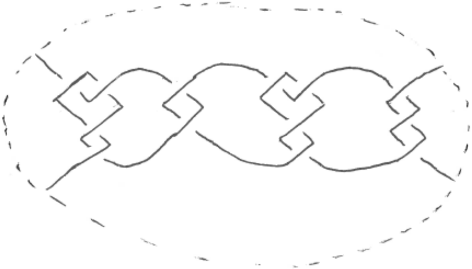


Montesinos Tangles

Definition

A tangle is **Montesinos** if it can be expressed as a horizontal or vertical joining of rational tangles.

Example: $(2/5) + (1/2) + (2/5) + (1/3)$

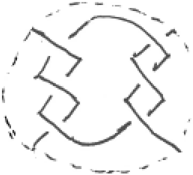


Generalized Montesinos Tangles

Definition

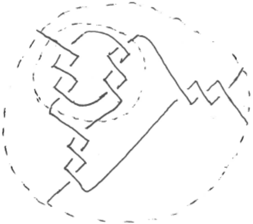
A **generalized Montesinos tangle** is a Montesinos tangle with endpoint twisting.

Example: Montesinos



$$\left(\frac{1}{-3}\right) + \left(\frac{1}{-3}\right)$$

Example: Generalized



$$\left(\frac{1}{-3}\right) + \left(\frac{1}{-3}\right) \circ (-3, -3, 0)$$

Database of 2-String Tangles

Prototype of database of 2-string tangles:

- http://www.nick-connolly.com/tangles
- rational tangles up to 9 crossings
- generalized Montesinos tangles up to 11 crossings

Algebraic Tangles

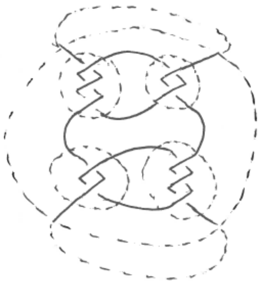
Definition

A tangle is **algebraic** if it can be expressed with a finite combination of sums and products of rational tangles.

Example:

$$((1/3)+(1/2))*((1/2)+(1/3))$$

Example of a non-Montesinos algebraic tangle.

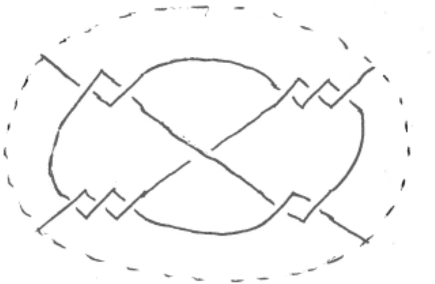


Non-Algebraic Tangles

Definition

A tangle is **non-algebraic** if it cannot be expressed in terms of a finite combination of sums and products of rational tangles.

Example:



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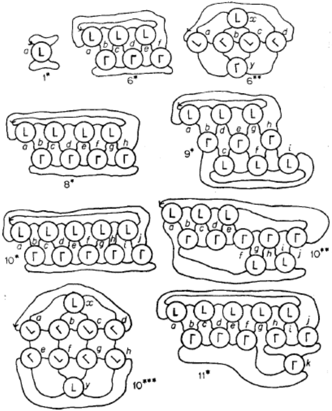
Recall: Conway's Tabulation[3]



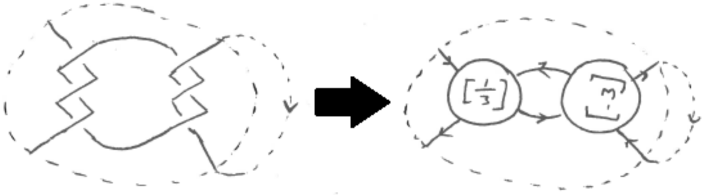
[3] J. H. Conway. An enumeration of knots and lints, and some of their algebraic properties. In *Computational Problems in Abstract Algebra (Proc. Conf., Oxford 1967)*, pages 329-358. Pergamon, Oxford, 1970

Adapting Conway's Idea

Observations:

- Rational tangles are well-behaved and easy to describe.
- Rational tangles "live inside" non-rational tangles.

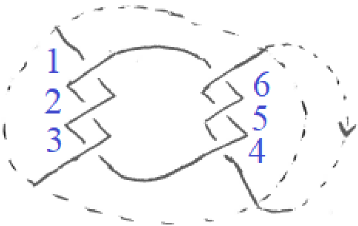
Idea: Can we use rational subtangles rather than crossings?



Recall: Planar Diagram Code

Example: $T = (1/3) + (1/3)$

$+$	$2b$	$3a$	$6d$	$2c$
$+$	$3b$	$1a$	$1d$	$3c$
$+$	$1b$	$2a$	$2d$	$4c$
$+$	$5b$	$6a$	$3d$	$5c$
$+$	$6b$	$4a$	$4d$	$6c$
$+$	$4b$	$5a$	$5d$	$1c$

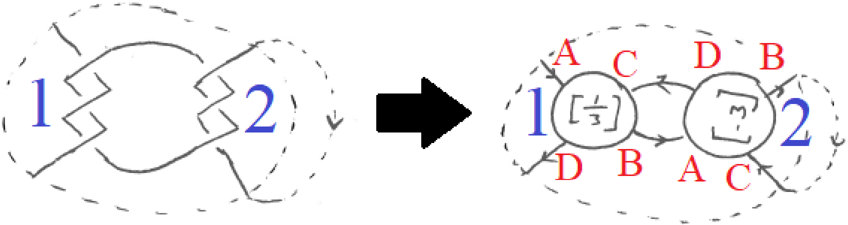


Tangles are described by their crossing connections.

A Generalized Planar Diagram Code

Example: $T = (1/3) + (1/3)$

$(1/3)$	1 ep	$2A$	$2D$	1 ep	$(\mathrm{P}:1)$	TwiVec:	$(1\ \ 2\ \ 0)$
$(3/-1)$	$1B$	2 ep	2 ep	$1C$	$(\mathrm{P}:1')$	TwiVec:	(-3)



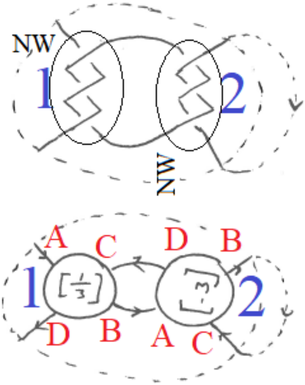
Tangles are described by their subtangle connections.

Generalized PD Code: Subtangle Labeling Convention

New Rules for Subtangles:

- Each subtangle has two strings: $S1$ and $S2$.
- Each corner matches a string entrance or exit.
- Enter/Exit Labeling rule:
  - $A = S1$ entrance
  - $B = S1$ exit
  - $C = S2$ entrance
  - $D = S2$ exit
- Identify A as NW.
- Denote subtangle by corresponding rational p/q.

Example: $T = (1/3) + (1/3)$

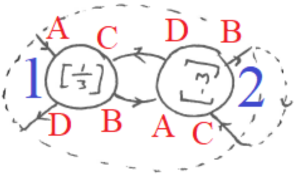


Generalized PD Code: Additional Information

Row Information:
(one subtangle per row)

- Rational number p/q
- Connection information
- Subtangle parity
  - internal string direction
- Rational twist vector

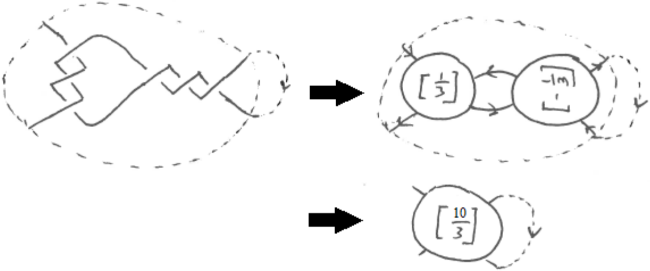
Example: $T = (1/3) + (1/3)$



$$\underbrace{\begin{matrix}(1/3)\\(3/-1)\end{matrix}}_{p/q}\quad \underbrace{\begin{matrix}1\text{ ep} & 2A & 2D & 1\text{ ep}\\ 1B & 2\text{ ep} & 2\text{ ep} & 1C\end{matrix}}_{\text{connections}}\quad \underbrace{\begin{matrix}(\mathrm{P}:1)\\(\mathrm{P}:1')\end{matrix}}_{\text{parity}}\quad \underbrace{\begin{matrix}\text{TwiVec:}(1\ \ 2\ \ 0)\\ \text{TwiVec:}\ \ (-3)\end{matrix}}_{\text{twist vector}}$$

Generalized PD Code: Rational Example

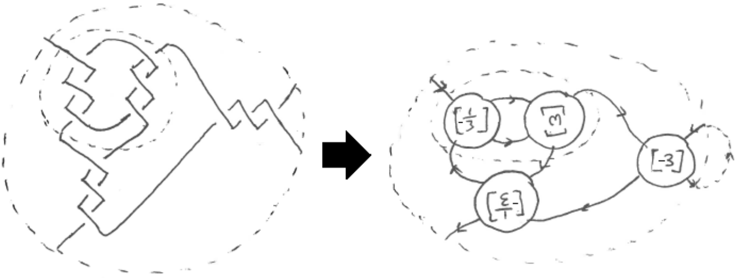
Example: $T = (1/3) + (3/1)$



(10/3) 1 ep 1 ep 1 ep 1 ep (P : 0) TwiVec: (1 2 3)

Generalized PD Code: Generalized Montesinos Example

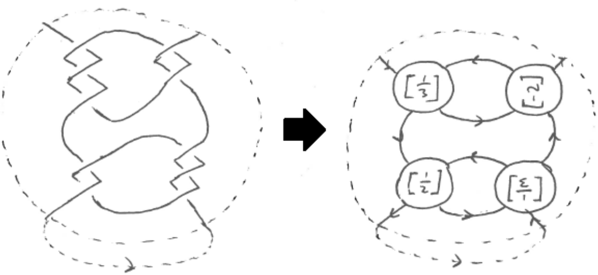
Example: $T = ((1/-3) + (1/-3)) \circ (-3, -3, 0)$



$(1/-3)$	4 ep	$2A$	$4B$	$2C$	$(P : 1')$	TwiVec:	$(-1 \quad -2 \quad 0)$
$(3/1)$	$1B$	$3A$	$1D$	$4C$	$(P : 1)$	TwiVec:	(3)
$(3/-1)$	$2B$	3 ep	3 ep	$4A$	$(P : 1)$	TwiVec:	(-3)
$(1/-3)$	$3D$	$1C$	$2D$	1 ep	$(P : 1')$	TwiVec:	$(-1 \quad -2 \quad 0)$

Generalized PD Code: Algebraic Example

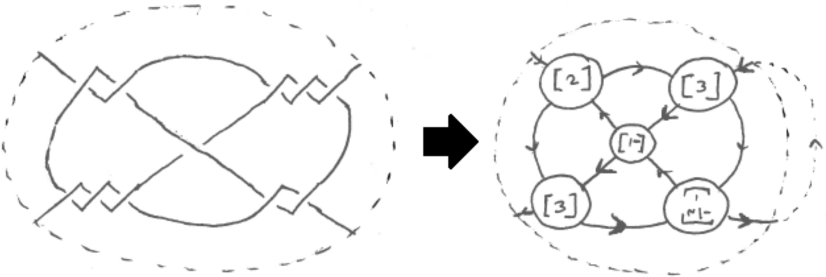
Example: $T = ((1/3) + (1/2)) * ((1/2) + (1/3))$



$(1/3)$	2 ep	$2A$	$2B$	$3A$	(P : 1)	TwiVec:	$(1\ \ 2\ \ 0)$
$(2/-1)$	$1B$	$1C$	$4D$	1 ep	(P : $0'$)	TwiVec:	(-2)
$(1/-2)$	$1D$	4 ep	$4B$	$4C$	(P : ∞')	TwiVec:	$(-1\ \ -1\ \ 0)$
$(1/3)$	3 ep	$3C$	$3D$	$2C$	(P : 1)	TwiVec:	$(1\ \ 2\ \ 0)$

Generalized PD Code: Non-Algebraic Example

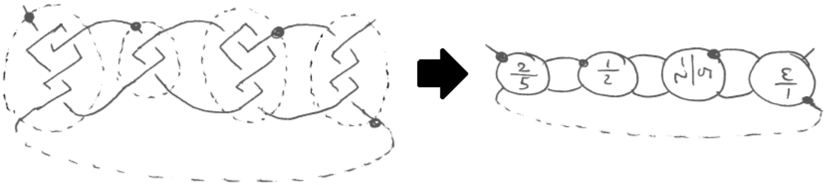
Example:



$(2/1)$	5 ep	$2A$	$4B$	$5A$	$(\mathrm{P}:0)$	TwiVec:	(2)
$(3/1)$	$1B$	$3A$	3 ep	$4C$	$(\mathrm{P}:1)$	TwiVec:	(3)
$(1/-2)$	$2B$	$4A$	$5B$	2 ep	$(\mathrm{P}:\infty)$	TwiVec:	$(-1\ \ -1\ \ 0)$
$(1/-1)$	$3B$	$1C$	$2D$	$5C$	$(\mathrm{P}:1')$	TwiVec:	(-1)
$(3/1)$	$1D$	$3C$	$4D$	1 ep	$(\mathrm{P}:1)$	TwiVec:	(3)

Original Labeling: Enter/Exit

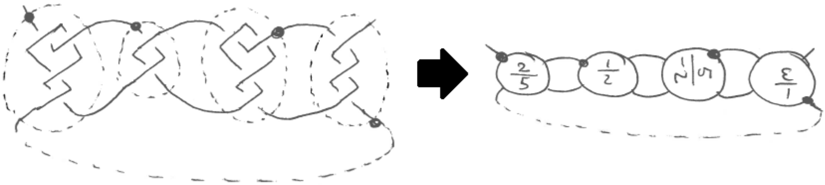
Example: $T = (2/5) + (1/2) + (2/5) + (1/3)$



$(2/5)$	3 ep	$2A$	$2B$	3 ep	(P:0)	TwiVec:	$(2\ \ 2\ \ 0)$
$(1/2)$	$1B$	$1C$	$4B$	$4C$	(P:∞')	TwiVec:	$(1\ \ 1\ \ 0)$
$(1/3)$	1 ep	$4A$	$4D$	1 ep	(P:1)	TwiVec:	$(1\ \ 2\ \ 0)$
$(5/-2)$	$3B$	$2C$	$2D$	$3C$	(P:∞)	TwiVec:	$(-1\ \ -1\ \ -2)$

Alternate Labeling: Compass

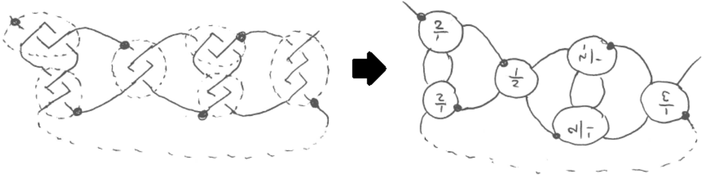
Example: $T = (2/5) + (1/2) + (2/5) + (1/3)$



(2/5)	3 ep	2 NW	2 SW	3 ep	(P:0)	TwiVec:	$(2 \;\; 2 \;\; 0)$
(1/2)	1 NE	4 SW	4 SE	1 SE	$(P{:}\infty')$	TwiVec:	$(1 \;\; 1 \;\; 0)$
(1/3)	1 ep	4 NE	4 NW	1 ep	(P:1)	TwiVec:	$(1 \;\; 2 \;\; 0)$
$(5/-2)$	3 SE	3 NE	2 SE	2 NE	$(P{:}\infty)$	TwiVec:	$(-1 \;\; -1 \;\; -2$

Integer Subtangle Planar Diagram Code

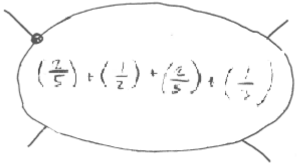
Example: $T = (2/5) + (1/2) + (2/5) + (1/3)$



$(2/1)$	4 ep	2 NW	3 SW	3 SE	$(\mathrm{P}{:}0)$
$(1/2)$	1 NE	5 SW	6 NW	3 NW	$(\mathrm{P}{:}\infty')$
$(1/2)$	2 SW	4 ep	1 SW	1 SE	$(\mathrm{P}{:}\infty)$
$(1/3)$	3 ep	6 SW	5 NW	1 ep	$(\mathrm{P}{:}1)$
$(1/-2)$	4 SE	6 SE	6 NE	2 NE	$(\mathrm{P}{:}\infty)$
$(2/-1)$	2 SE	5 SE	5 NE	4 NE	$(\mathrm{P}{:}0)$

Algebraic Subtangle Planar Diagram Code

Example: $T = (2/5) + (1/2) + (2/5) + (1/3)$



$(2/5) + (1/2) + (2/5) + (1/3)$ 1 ep 1 ep 1 ep 1 ep (P:∞)

Benefits of Programming with New Notation

In my research, I have developed a program that can:

- compute all generalized PD codes for given tangle;
- determine the algebraic construction;
- determine whether the diagram is internally canonical.

impress class with sample program output

Table of Contents

The Formal Definition of Graph

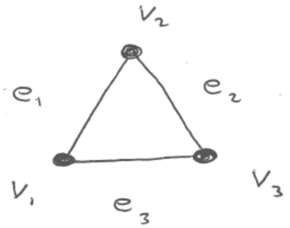
Definition

A **graph** $G = (V, E)$ consists of two things:

- a collection of **vertices** V
- a collection of **edges** E

Example: $G = (V, E)$

$$\begin{aligned} V &= \{v_1, v_2, v_3\} \\ E &= \{e_1, e_2, e_3\} \end{aligned}$$

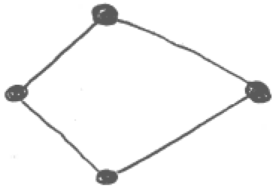


Simple Graphs vs. Multi-Graphs

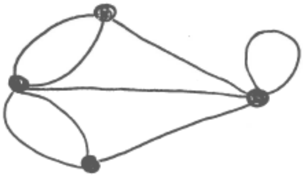
Definition

A graph is said to be **simple** if no two vertices are joined by more than one edge. A graph which is not simple is a **multi-graph**.

Example: Simple Graph



Example: Multi-Graph

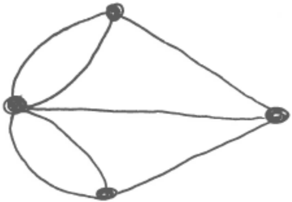


Planar Graphs

Definition

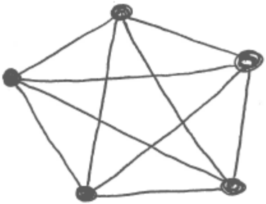
A graph is **planar** if it can be drawn with no edges intersecting.

Example: Planar Graph



Konigsberg Graph

Example: Non-Planar Graph



K_5

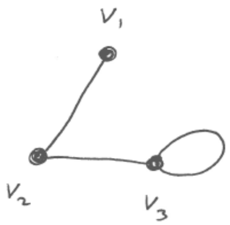
Vertex Degrees

Definition

The **degree** of a vertex v, denoted $\delta(v)$, is the number of edges connected to it. Loops are counted twice.

Example:

$$\begin{aligned}\delta(v_1) &= 1\\ \delta(v_2) &= 2\\ \delta(v_3) &= 3\end{aligned}$$



Definition

A graph in which each vertex has degree k is called k-**regular**.

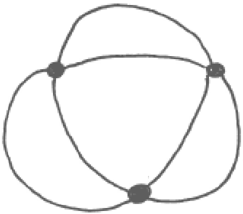
The Graph of a Knot Diagram

Any knot diagram can be turned into a graph.

- The crossings become vertices.
- The strands between crossings become edges.

Example: Trefoil Diagram

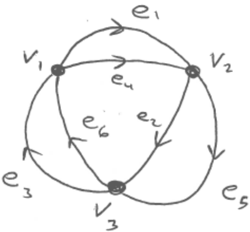
Example: Trefoil Graph



Knot Orientation and Eulerian Circuits

Observe that a choice of orientation on a knot diagram induces an Eulerian circuit on the corresponding knot graph.

Example: Oriented Trefoil



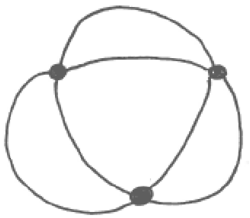
$$s = [v_1, e_1, v_2, e_2, v_3, e_3, v_1, e_4, v_2, e_5, v_3, e_6, v_1]$$

Properties of a Knot Diagram Graph

Properties:

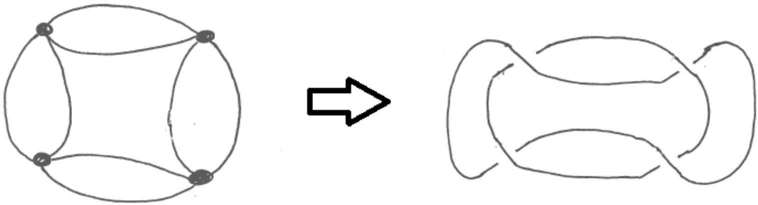
- The graph is **connected**.
- The graph is **planar**.
- The graph is **4-regular** (each vertex has degree 4).
- The graph contains an **Eulerian circuit**.

Graph:



Reverse Question: Building Knot Diagrams from Graphs

Any connected, planar, 4-regular graph can be realized as a link.



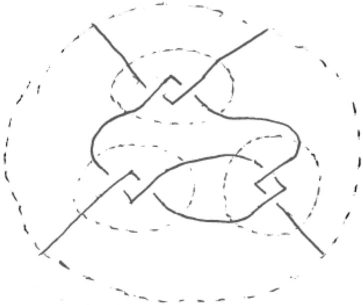
Graph Diagram

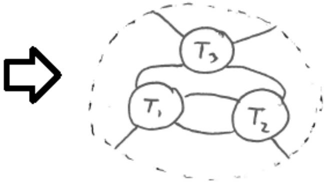
A graph with n vertices can be realized by 2^n diagrams.

Describing Tangles with Graphs

The generalized planar diagram code describes a tangle through joined subtangles; this description is graph theoretic!

Example: $T_3 * (T_1 + T_2)$



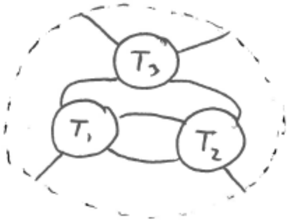


Subtangle Graphs

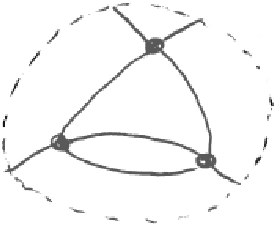
We can also model these subtangle combinations with a graph.

- The subtangles becomes vertices.
- The strands between subtangles become edges.

Example: Tangle Diagram



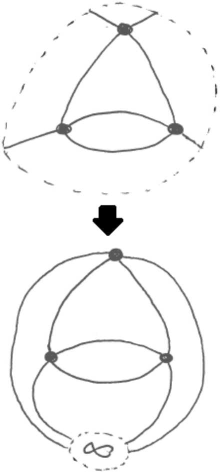
Example: Subtangle "Graph"



Properties of a Subtangle Diagram Graph

Properties:

- The graph is **connected**.
- The graph is **planar**.
- The graph is **4-regular** (each vertex has degree 4).
- The graph contains an **Eulerian circuit**.
- The endpoints form an extra **vertex at ∞**.

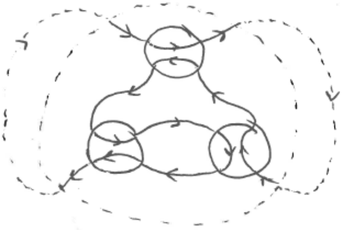


Eulerian Circuits and Subtangle Parities

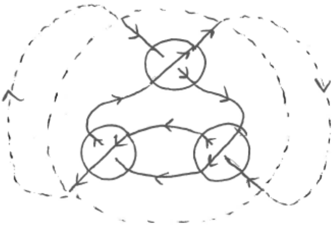
Theorem

A choice of Eulerian circuit in a subtangle diagram graph uniquely induces a parity on each subtangle component.

Example: Parities 0, 0, ∞



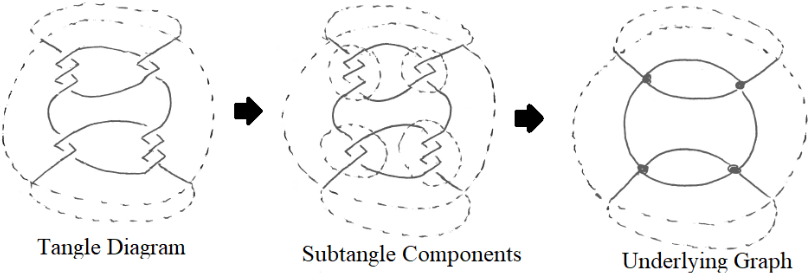
Example: Parities 1, 1, 1



Finding possible parities is equivalent to finding Eulerian circuits.

Reversing the Question

Every tangle diagram defines a graph with these properties.

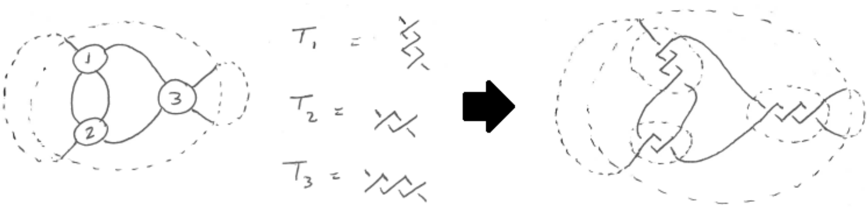


Tangle Diagram → Subtangle Components → Underlying Graph

Does every graph with these properties define a diagram?

Building a Diagram from a Graph

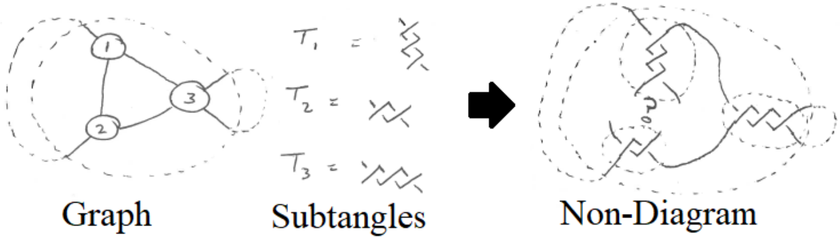
Given a graph with these properties, we can build up a configuration of subtangles. Filling these in gives a tangle diagram.



Graph Subtangles Diagram

Necessary Assumptions

Notice that we cannot connect subtangles without 4-regularity.



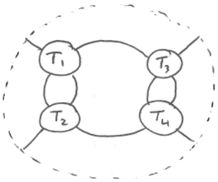
Can we add more restrictions to distinguish types of diagrams?

Algebraic Diagrams

Definition

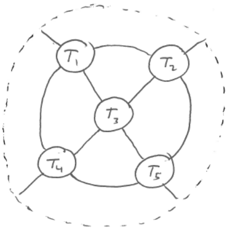
A subtangle diagram is **algebraic** if it can be built from sums and products of subtangles.

Example: Algebraic



$(T_1 * T_2) + (T_3 * T_4)$

Example: Non-Algebraic

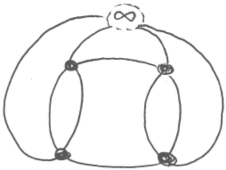


Tangle sums and products always join two endpoints at a time.

Graphs of Algebraic Diagrams

In the corresponding subtangle graph, there are two edges between added/multiplied subtangles (it's a **multi-graph**).

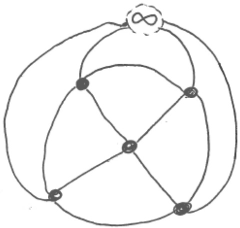
Example: Algebraic



$$(T_1 * T_2) + (T_3 * T_4)$$

Graph has multi-edges

Example: Non-Algebraic



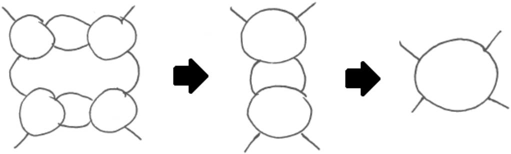
Graph is simple

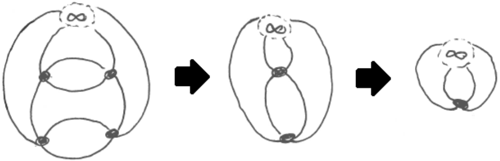
The Maximal Algebraic Subtangle Graph

Definition

The **maximal algebraic subtangle graph** of a tangle diagram is obtained by collapsing together any vertices with two connections.

Example:





Distinguishing Algebraic and Non-Algebraic Diagrams

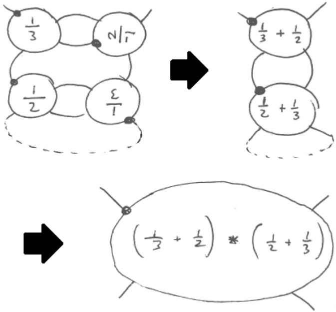
Theorem

A tangle is algebraic if and only if its **maximal algebraic subtangle graph** *consists of a single subgtangle vertex.*

Example: Algebraic Tangle

$((1/3)+(1/2))*((1/2)+(1/3))$

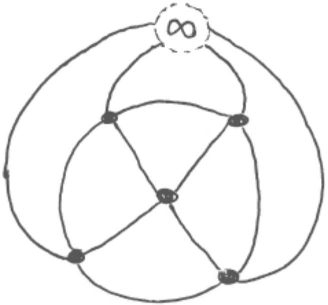
MASG:

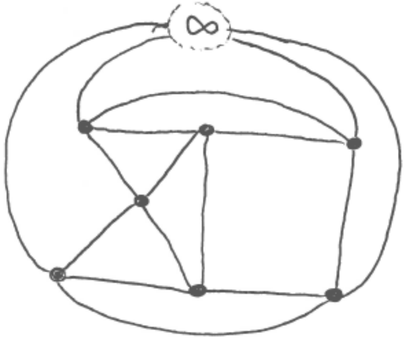


When is a Diagram Non-Algebraic?

Theorem

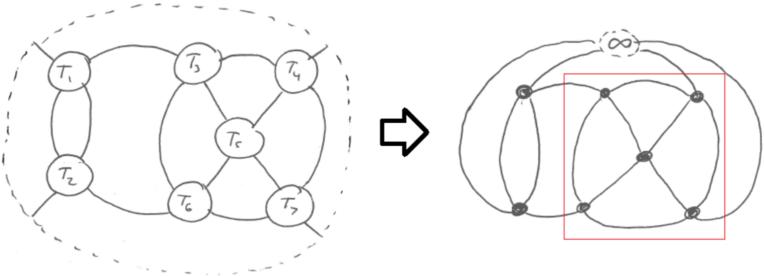
A **simple** *subtangle graph must be non-algebraic.*





Non-Algebraic Condition: Sufficient but not Necessary

Simple subtangle graphs must be non-algebraic; the converse does **not** hold. A non-algebraic subtangle graph isn't always simple.



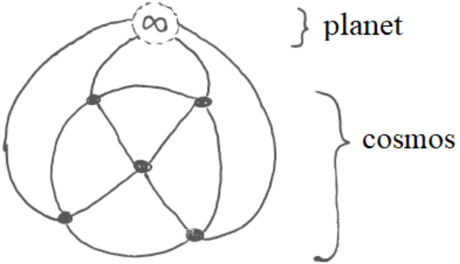
However, this graph contains a particular simple **subgraph**.

A New Type of Graph: Constellations

Definition

A $k+1$ **constellation** (C, p) consists of a graph C with $k+1$ vertices which is connected, planar, 4-regular, and **simple**, and a marked $k+1^{st}$ vertex p.

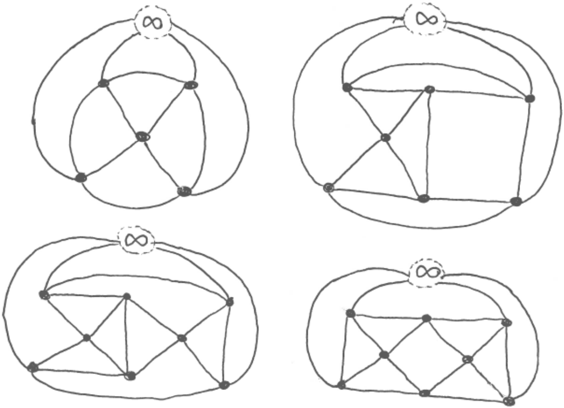
- The $k+1^{st}$ vertex p is called the **planet**.
- The induced subgraph $C^* = C - \{p\}$ is called the **cosmos**.



Constellation Examples

Definition

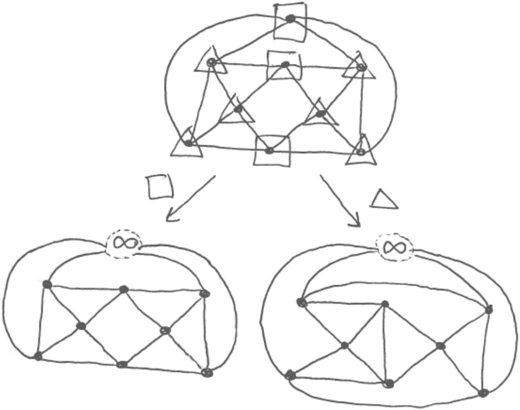
Two constellations are **equivalent** if they have isomorphic cosmos.



Changing Planets

Theorem

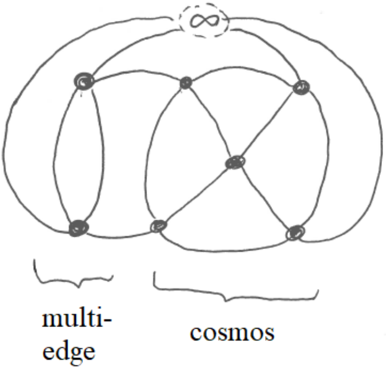
A different choice of planet may yield non-equivalent constellations.



Constellations and Non-Algebraic Subtangle Graphs

Theorem

A subtangle graph is non-algebraic if and only if it contains a subgraph which is isomorphic to the cosmos of some constellation.



Constellation Notation for Non-Algebraic Tangles

Definition

A non-algebraic tangle diagram can be described using sums and products of rational subtangles and constellations:

- use constellations in place of non-algebraic subtangles;
- list out the algebraic subtangle components;
- order tangles by encounter and indicate parity.

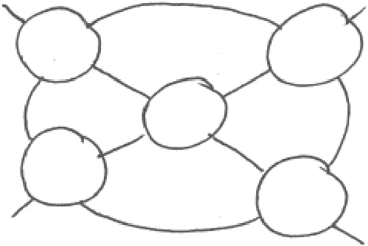
Example:

$$\underbrace{C^{(1)}_{5+1}}_{\text{constellation}} \quad \left((2/1)^0, \underbrace{(3/1)^1}_{\text{subtangle}}, (1/-2)^\infty, (1/-1)^{1'}, (3/1)^1\right)$$

The subtangles come from the algebraic planar diagram code.

Reconstructing from Notation: Cosmos

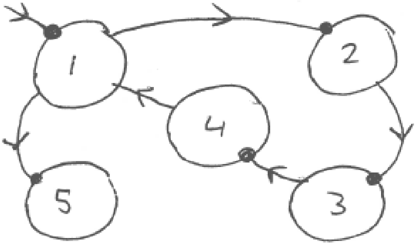
$$\underbrace{C_{5+1}^{(1)}}_{\text{constellation}} \quad \left((2/1)^0, (3/1)^1, (1/-2)^\infty, (1/-1)^{1'}, (3/1)^1\right)$$



This tangle uses the cosmos of the $5+1$ constellation.

Reconstructing from Notation: Component Order

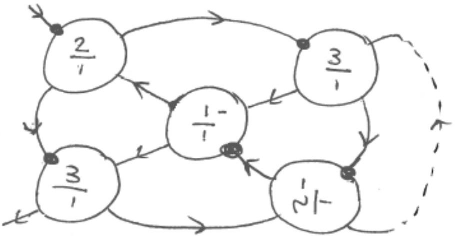
$$C^{(1)}_{5+1}\left(\underbrace{(2/1)^0}_{1},\underbrace{(3/1)^1}_{2},\underbrace{(1/-2)^\infty}_{3},\underbrace{(1/-1)^{1'}}_{4},\underbrace{(3/1)^1}_{5}\right)$$



The parity of each component forces the location of the next component. In this example, the ordered parities are $(0, 1, \infty, 1', 1)$.

Reconstructing from Notation: Filling in Components

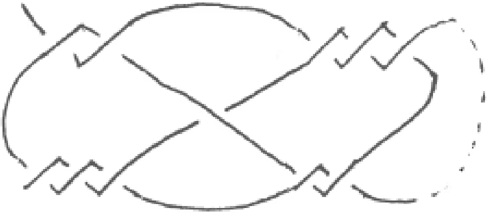
$$C_{5+1}^{(1)}\left(\underbrace{(2/1)^0}_{1},\underbrace{(3/1)^1}_{2},\underbrace{(1/-2)^\infty}_{3},\underbrace{(1/-1)^{1'}}_{4},\underbrace{(3/1)^1}_{5}\right)$$



Insert appropriate subtangle construction into each component.

Reconstructing from Notation: Drawing Subtangles

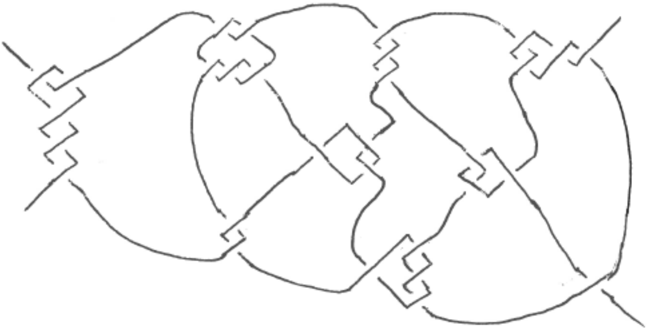
$$C^{(1)}_{5+1}\left((2/1)^0, (3/1)^1, (1/-2)^\infty, (1/-1)^{1'}, (3/1)^1\right)$$



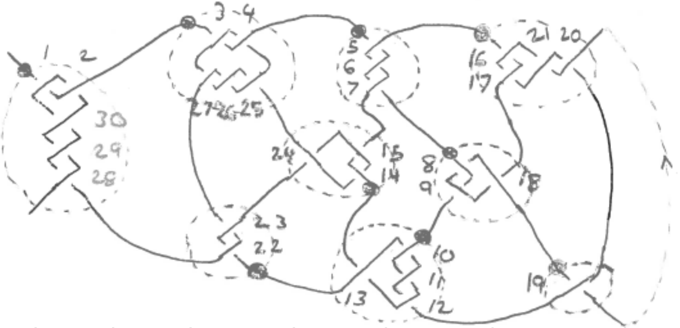
Finally, draw in algebraic subtangles given by construction.

Example: Scary Non-Algebraic Diagram

How to describe this diagram?

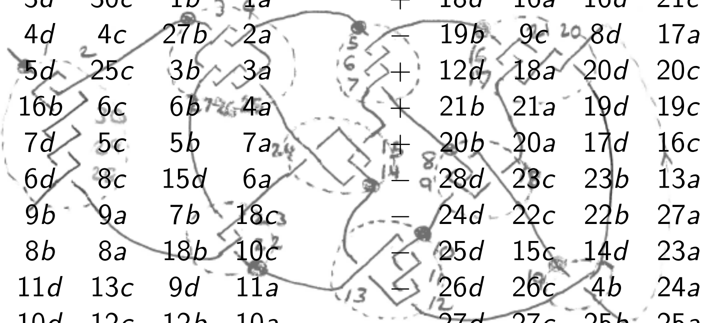


Example: Gauss Code



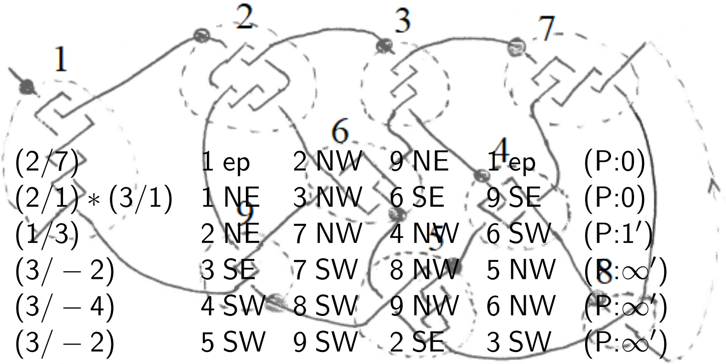
(|-b1-a2-b3-a4-b5-a6-b7+a8+b9-a10-b11-a12-b13+a14+b15-a7
-b6-a5+b16+a17-b18+a9+b8-a18+b19(|)+a20+b21+a16+b17
+a21+b20+a19-b12-a11-b10-a13-b22-a23-b24+a15+b14-a24
-b25-a26-b27-a3-b4-a25-b26-a27-b23-a22-b28-a29-b30-a1-b2-a30
-b29-a28|)

Example: Planar Diagram Code



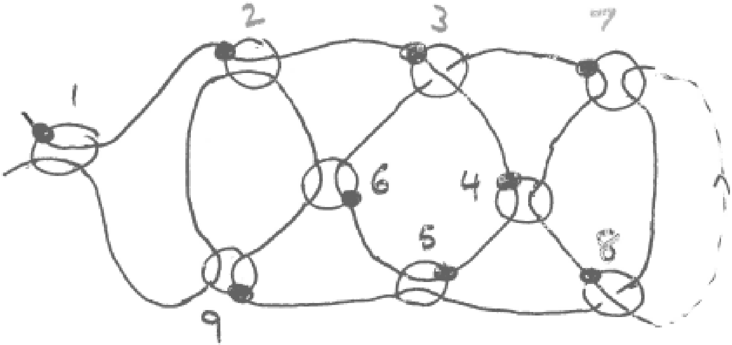
$-$	$2d$	$2c$	$30b$	$28a$	$+$	$17b$	$5a$	$21d$	$17c$
$-$	$3d$	$30c$	$1b$	$1a$	$+$	$18d$	$16a$	$16d$	$21c$
$-$	$4d$	$4c$	$27b$	$2a$	$-$	$19b$	$9c$	$8d$	$17a$
$-$	$5d$	$25c$	$3b$	$3a$	$+$	$12d$	$18a$	$20d$	$20c$
$-$	$16b$	$6c$	$6b$	$4a$	$+$	$21b$	$21a$	$19d$	$19c$
$-$	$7d$	$5c$	$5b$	$7a$	$+$	$20b$	$20a$	$17d$	$16c$
$-$	$6d$	$8c$	$15d$	$6a$	$-$	$28d$	$23c$	$23b$	$13a$
$+$	$9b$	$9a$	$7b$	$18c$	$-$	$24d$	$22c$	$22b$	$27a$
$+$	$8b$	$8a$	$18b$	$10c$	$-$	$25d$	$15c$	$14d$	$23a$
$-$	$11d$	$13c$	$9d$	$11a$	$-$	$26d$	$26c$	$4b$	$24a$
$-$	$10d$	$12c$	$12b$	$10a$	$-$	$27d$	$27c$	$25b$	$25a$
$-$	$13d$	$11c$	$11b$	$19a$	$-$	$23d$	$3c$	$26b$	$26a$
$-$	$22d$	$14c$	$10b$	$12a$	$-$	$1d$	$29c$	$29b$	$22a$
$+$	$15b$	$15a$	$13b$	$24c$	$-$	$30d$	$28c$	$28b$	$30a$
$+$	$14b$	$14a$	$24b$	$7c$	$-$	$29d$	$1c$	$2b$	$29a$

Example: Algebraic Planar Diagram Code



(2/7)	1 ep	2 NW	9 NE	1 ep	(P:0)
$(2/1) * (3/1)$	1 NE	3 NW	6 SE	9 SE	(P:0)
(1/3)	2 NE	7 NW	4 NW	6 SW	$(P{:}1')$
$(3/-2)$	3 SE	7 SW	8 NW	5 NW	$(P{:}\infty')$
$(3/-4)$	4 SW	8 SW	9 NW	6 NW	$(P{:}\infty')$
$(3/-2)$	5 SW	9 SW	2 SE	3 SW	$(P{:}\infty')$
(5/2)	3 NE	8 ep	8 NE	4 NE	$(P{:}\infty')$
(1/1)	4 SE	7 SE	7 ep	5 NE	(P:1)
(1/2)	5 SE	1 SE	2 SW	6 NE	$(P{:}\infty)$

Example: Algebraic Subtangle Components by Parity

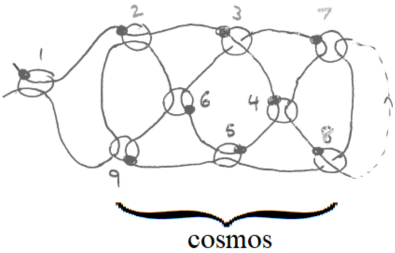


APD Gauss Code : $[1, 2, 3, 4, 5, 6, 3, 7, 4, 8, |7, 8, 5, 9, 6, 2, 9, 1]$

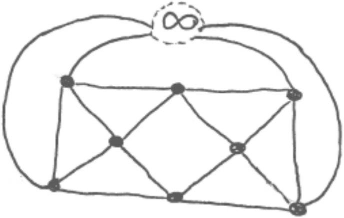
Parity Vector : $(0, 0, 1', \infty', \infty', \infty', \infty', 1, \infty)$

Example: Identifying Cosmos of 8+1 Constellation

Subtangle Graph

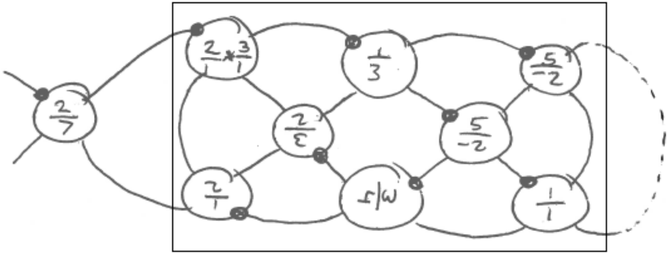


8+1 Constellation



$$\begin{array}{rcl} \text{APD Gauss Code} & : & [1,2,3,4,5,6,3,7,4,8,|7,8,5,9,6,2,9,1] \\ \text{Parity Vector} & : & (0,0,1',\infty',\infty',\infty',\infty',1,\infty) \end{array}$$

Example: Constellation Notation



algebraic + non-algebraic

$$T = (2/7) + C^{(1)}_{8+1}\Big((2/1) * (3/1)^{0}, (1/3)^{1'}, (3/-2)^{\infty'}, (3/-4)^{\infty'},$$
$$(3/-2)^{\infty'}, (5/2)^{\infty'}, (1/1)^{1}, (1/2)^{\infty}\Big)$$

$$\text{Parity Vector}: (0, 0, 1', \infty', \infty', \infty', \infty', 1, \infty)$$

Constellation Notation Summary

Benefits:

- compact notation for non-algebraic diagrams
- combines with sum/product notation
- construction intuitive to visualize

Drawbacks:

- always assumes internally canonical components
- only helpful for known constellations
- classifies non-algebraic diagrams, **but not tangles**

Major Advantage:

preferring canonical components eliminates many redundancies