An Introduction to Rational Tangles and Some of Their Generalizations

Nicholas Connolly

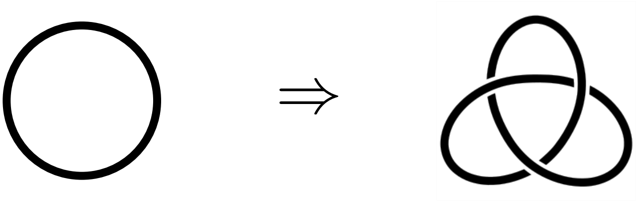
Kenyon College Math Monday

February 7, 2022

Table of Contents

Knot Theory and Knots

Knot theory concerns the study of **mathematical knots**.

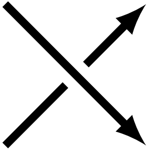


A **knot** is any *homeomorphic embedding* of the 1-dimensional circle S^1 into 3-dimensional space $\mathbb{R}^3$.

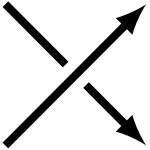
Knot Diagrams and Orientation

A **knot diagram** is the projected shadow of a knot onto a 2-dimensional surface with the crossings identified.

An **orientation** on a knot is a choice of direction for the string.



Positive Crossing

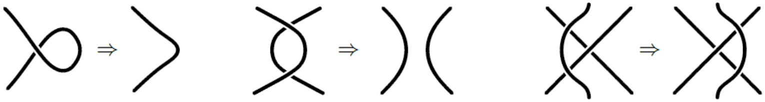


Negative Crossing

An orientation on a knot diagram induces a **crossing sign** (positive or negative) for each crossing in a diagram.

Reidemeister Moves

Two knot diagrams describe **equivalent** knots if they are related by a sequence of *Reidemeister moves*.

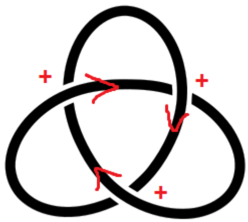


(a) RI move (b) RII move (c) RIII move

Knot diagrams are usually represented using **diagram notations**: a sequence of symbols describing the strings and crossings.

Example: Diagram Notations for the Trefoil

- Dowker-Thistlewaite Code:
  $-4 \quad -6 \quad -2$
- Gauss Code:
  $a1+\ b2+\ a3+\ b1+\ a2+\ b3+$
- Planar Diagram Code:
  $X_{4,2,5,1}X_{2,6,3,5}X_{6,4,1,3}$
- Coloring Matrix:
  $$\begin{bmatrix} -2 & 1 & 1 \\ 1 & 1 & -2 \\ 1 & -2 & 1 \end{bmatrix}$$



- Ewing-Millett Code:
  $$\begin{matrix} + & 2b & 3a & 3d & 2c \\ + & 3b & 1a & 1d & 3c \\ + & 1b & 2a & 2d & 1c \end{matrix}$$

Knot Invariants

A **knot invariant** is any *"quantity"* defined for a knot which is the same for equivalent knots.

Examples of invariants for the trefoil[1]:

- Minimal crossing number: 3
- Unknotting number: 1
- Bridge index: 2
- Jones polynomial: $V(q) = -q^{-4} + q^{-3} + q^{-1}$

Non-equivalent knots can be distinguished using invariants.

[1] The Knot Atlas: `http://katlas.math.toronto.edu/wiki/Trefoil`

Application: DNA Topology

Beyond pure mathematics, knots have been used to model protein folding and molecular interactions involving DNA.

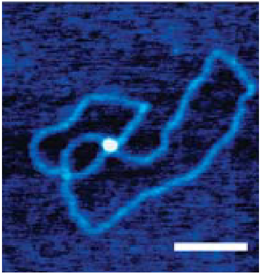


Figure: Protein DNA Complex (AFM)[2]

[2]Vetcher, Alexandre A., et al. "DNA topology and geometry in Flp and Cre recombination." Journal of molecular biology 357.4 (2006).

Table of Contents

n-String Tangles

An *n*-**string tangle** is an embedding of *n* disjoint string-segments into the interior of a 3-dimensional ball.

The strings' endpoints are embedded on the surface of the ball.

(a) 2-string tangle

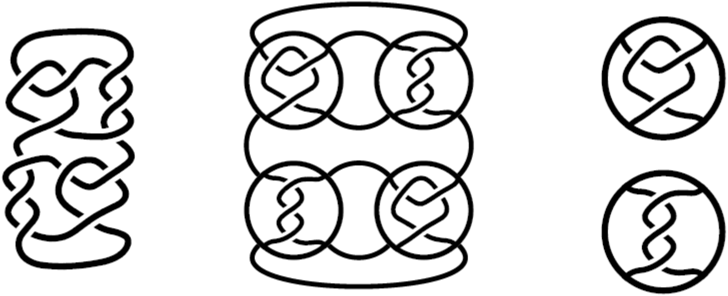
(b) 3-string tangle

(c) Non-tangle with closed loops

My research is concerned with the study of **2-string tangles**.

Relationship between Tangles and Knots

Tangles can be thought of as the *building blocks* of knots.

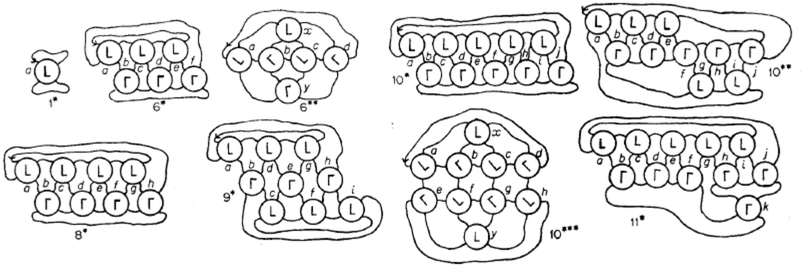


(a) Knot diagram (b) Knot building blocks (c) Parts of a knot

Conway's Planar Graphs

Theorem (Conway[3])

All knots with ≤ 11 *crossings can be described using one of these planar diagrams, where vertices are replaced with* *rational tangles*.



[3] J. H. Conway. An enumeration of knots and links, and some of their algebraic properties. In *Computational Problems in Abstract Algebra (Proc. Conf., Oxford 1967)*, pages 329-358. Pergamon, Oxford, 1970

Conventions for Tangle Diagrams

The *boundary ball* of a tangle is represented by a *boundary circle* in a tangle diagram; the four endpoints of the strings are arranged around the four corners of this circle: NW, NE, SE, SW.

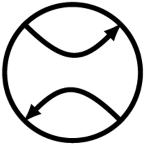
(a) Tangle diagram with endpoints labeled

(b) Non-equivalent tangle diagram after rotation

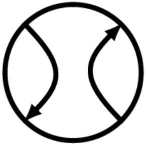
Endpoints must be fixed; rotating or flipping a tangle diagram can change the equivalence class of the tangle (unlike knots).

Tangle Parities and Diagram Orientations

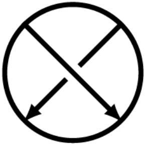
The **parity** of a tangle diagram is determined by how the strings connect to the endpoints; there are 3 (non-oriented) parities.



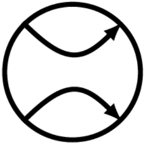
(a) Parity 0



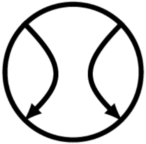
(b) Parity ∞



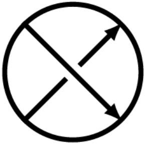
(c) Parity 1



(d) Parity $0'$



(e) Parity ∞'



(f) Parity $1'$

An **orientation** on a tangle diagram is a choice of direction for each string; by convention, the NW corner points inward.

Tangle Closures

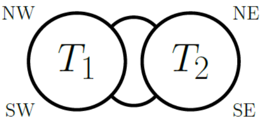
A tangle diagram can be **closed** into a knot or link diagram by connecting the endpoints together; there are two standard closures.

(a) Numerator Closure: $N(T)$ (b) Denominator Closure: $D(T)$

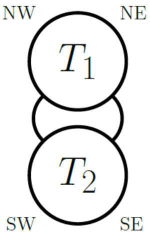
Whether a closure yields a knot or a link depends on the parity.

Tangle Algebra and Subtangles

Tangles can be stacked together like blocks to build larger tangles. The joined tangles become **subtangles** of the larger diagram.



(a) Tangle Sum: $T_1 + T_2$

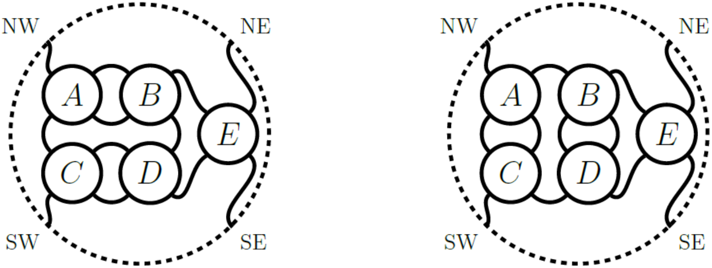


(b) Tangle Product: $T_1 * T_2$

These operations are known as tangle **sum** and **product**. Combinations which introduce a closed loop are not permitted.

Algebraic Construction Formulas

An **algebraic construction formula** describes how to build up a tangle diagram using sums and products of subtangles.



(a) $((A + B) * (C + D)) + E$ **(b)** $((A * C) + (B * D)) + E$

These are an example of a **subtangle decomposition**.

Hierarchy of 2-String Tangle Types

$$
\begin{array}{c}
\{\text{arbitrary 2-string tangles}\} \\
\cup\!\mid \\
\{\text{algebraic tangles}\} \\
\cup\!\mid \\
\{\text{generalized Montesinos tangles}\} \\
\cup\!\mid \\
\{\text{Montesinos tangles}\} \\
\cup\!\mid \\
\{\text{rational tangles}\} \\
\cup\!\mid \\
\{\text{integer tangles}\}
\end{array}
$$

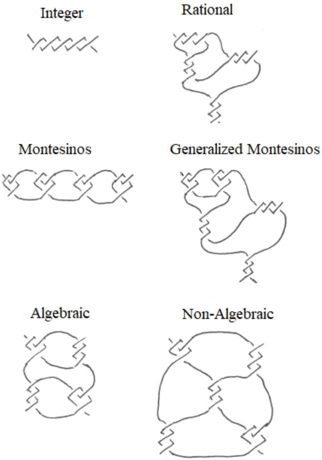
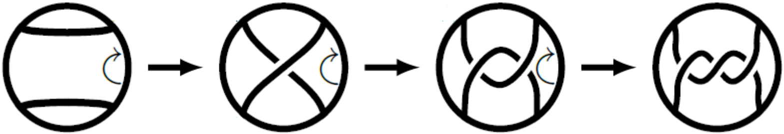


Table of Contents

Special Case: Integer Tangles

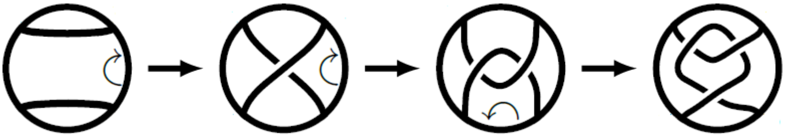
An **integer tangle** is defined by any sequence of horizontal or vertical twists (of the same *twist sign*).



The twist sign of a tangle matches the slope of the overstrands.

Intuitive Definition of a Rational Tangle

A **rational tangle** is defined by any combination of both horizontal and vertical twists (not necessarily of the same sign).



Twists may be applied on any side (top, right, bottom, left).

rational tangle examples

non-rational tangle

(Unwieldy) Inductive Definition of a Rational Tangle

Definition (Kauffman and Goldman[4])

For any sequence of integers $a_1, \cdots, a_n$, choose a sequence of integer tangles $S_{a_1}, \cdots, S_{a_n}$, where S_{a_i} is an integer tangle with $|a_i|$ twists. Define a tangle B_n by the following inductive construction.

1. $B_1 = S_{a_1}$.
2. For $k < n$:
   - If $S_{a_{k+1}}$ is horizontal, then either let $B_{k+1} = S_{a_{k+1}} + B_k$ or let $B_{k+1} = B_k + S_{a_{k+1}}$ (addition on left or right by $S_{a_{k+1}}$).
   - If $S_{a_{k+1}}$ is vertical, then either let $B_{k+1} = S_{a_{k+1}} * B_k$ or let $B_{k+1} = B_k * S_{a_{k+1}}$ (multiplication on top or bottom by $S_{a_{k+1}}$).

Any tangle B_n constructed by this algorithm is a **rational tangle** with n integer tangles.

---

[4] Jay R. Goldman and Louis H. Kauffman. Rational tangles. *Adv. in Appl. Math.*, 18(3):300-332, 1997

Example: Equivalent Rational Tangle Diagrams

Diagrams with different constructions may describe equivalent rational tangles. There always exists a preferred **canonical form**.

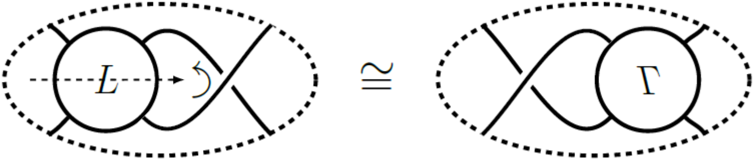
$\cong$

A rational tangle diagram in canonical form depends on:

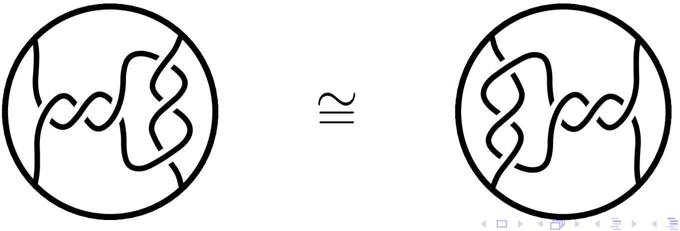
- the direction of twisting (right, bottom, left or top);
- the sign of the twists (positive or negative).

Shuffling Twists with Flypes

Twists can be moved across subtangles using **flypes**.



In a rational tangle diagram, all horizontal twists can be shuffled to the right, and all vertical twists can be shuffled to the bottom.



Canonical Form

A rational tangle diagram is said to be in **canonical form** provided

- it is constructed using only right and bottom twists;
- all twists have the same sign (positive or negative)

A canonical diagram is:

- alternating;
- minimal;
- unique;
- obtainable using Redeimesiter moves.

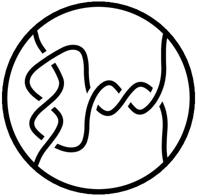
Theorem (Conway)

Every rational tangle admits a unique canonical diagram.

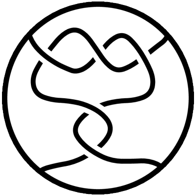
Twist Vectors

Every rational tangle diagram constructed using right and bottom twists (canonical or not) can be described using a **twist vector**.

$(5, 4, 3, 2, 1)$



$(1, 2, 3) \cong (3, 3)$



$(-3, -2, 0)$

- Entries represent alternately horizontal and vertical twists.
- The sign of each <u>non-zero</u> entry denotes the twist sign.
- The last entry always denotes a horizontal twist.
- Twist vectors are not unique*.

The Continued Fraction

Every twist vector defines a **continued fraction** which simplifies to some extended rational number $\frac{p}{q} \in \mathbb{Q} \cup \{\infty\}$.

$$\mathrm{CF}(n_1, \cdots, n_r) = n_r + \cfrac{1}{n_{r-1} + \cdots + \frac{1}{n_2 + \frac{1}{n_1}}} = \frac{p}{q}$$

$$\begin{aligned} &\mathrm{CF}(5,4,3,2,1) \\ &= 1 + \cfrac{1}{2 + \frac{1}{3 + \frac{1}{4 + \frac{1}{5}}}} \\ &= \frac{225}{157} \end{aligned}$$

Conway's Bijection

Theorem (Conway[5])

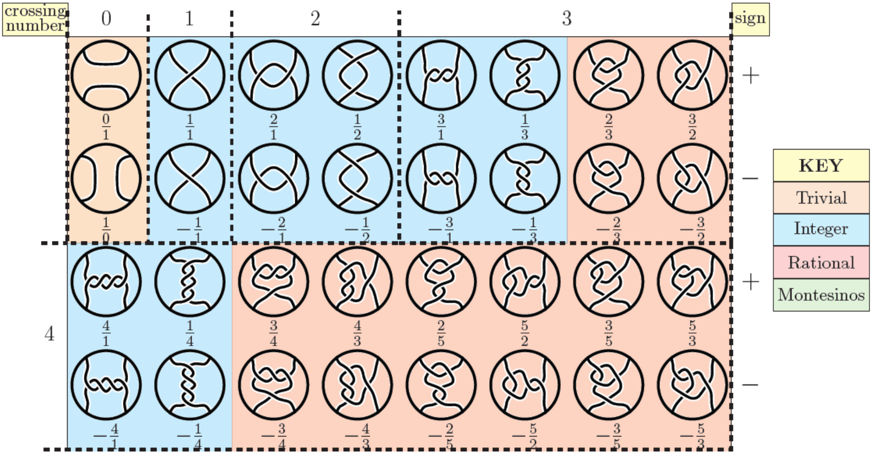
There exists a bijection between the rational tangles and the extended rational numbers defined by the continued fraction.

$$(-3,-3,3) \cong (3,2,1,2) \cong (1,2,2,1,2)$$

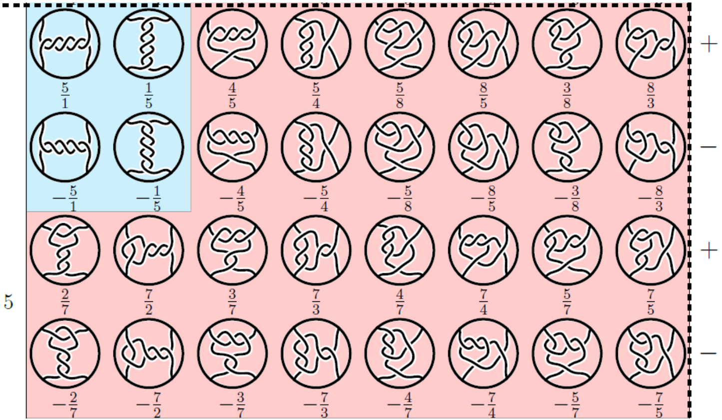
$$\mathrm{CF}(-3,-3,3) = \mathrm{CF}(3,2,1,2) = \mathrm{CF}(1,2,2,1,2) = \frac{27}{10}$$

[5] J. H. Conway. An enumeration of knots and links, and some of their algebraic properties. In *Computational Problems in Abstract Algebra (Proc. Conf., Oxford 1967)*, pages 329-358. Pergamon, Oxford, 1970

Example: Table of Small Rational Tangles (≤ 4 crossings)



Example: Table of Small Rational Tangles (5 crossings)



The Tanglenomicon[6]

One of the products of my doctoral research is the creation of a prototypical database of 2-string tangles!

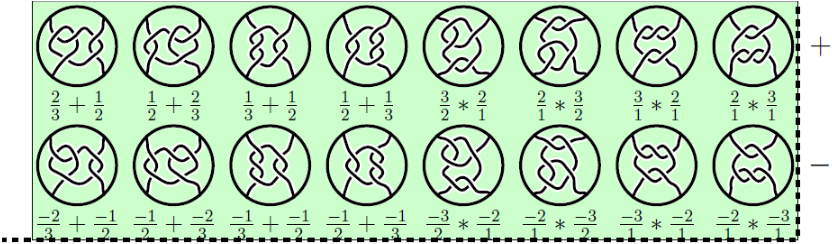
`http://www.nick-connolly.com/tangles_dev/tangles_rational.php`

[6] *That is knot thread which can eternal tie.*

Table of Contents

Example: Non-Rational 5-Crossing Tangles

A tangle is **non-rational** if it cannot be constructed using only twists; the smallest non-rational tangle has 5 crossings.

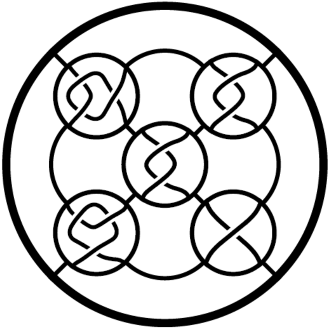


Algebraic and Non-Algebraic Tangles

A tangle is **algebraic** if it can be constructed using any combination of sums and products of rational tangles.

A tangle is **non-algebraic** if it cannot be constructed in this way.

$T \cong (\frac{1}{2} + \frac{5}{3}) * \frac{2}{5}$

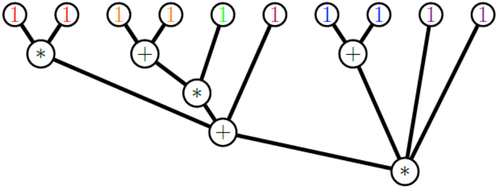


A Visualization of Algebraic Semi-Canonical Form

Algebraic tangle diagrams have a **semi-canonical form** that can be defined and visualized using *algebraic tangle trees*.

$T \cong \left(\frac{1}{2} + \frac{5}{3}\right) * \frac{2}{5}$

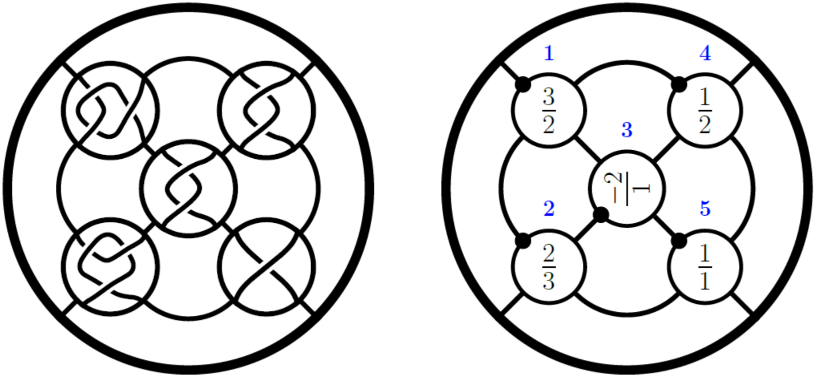
$$((1 * 1) + ((1 + 1) * 1) + 1) * (1 + 1) * 1 * 1$$



Equivalent right-veering reduced algebraic tangle tree $A^r(T)$

Subtangle Decompositions

Any tangle diagram (algebraic or not) can be decomposed into **maximal subtangles** of a certain type.

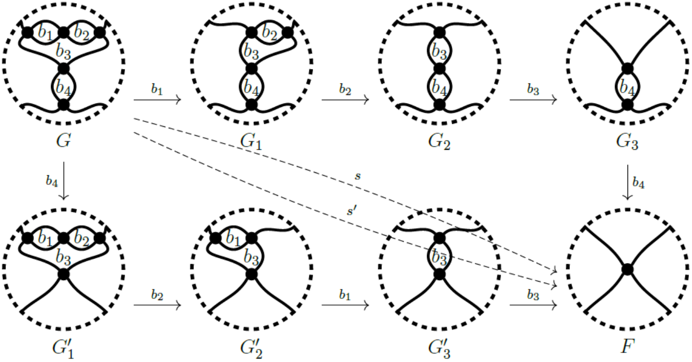


(a) Non-algebraic tangle diagram

(b) Maximal algebraic subtangle decomposition

Maximal Algebraic Subtangle Graphs (MASGs)

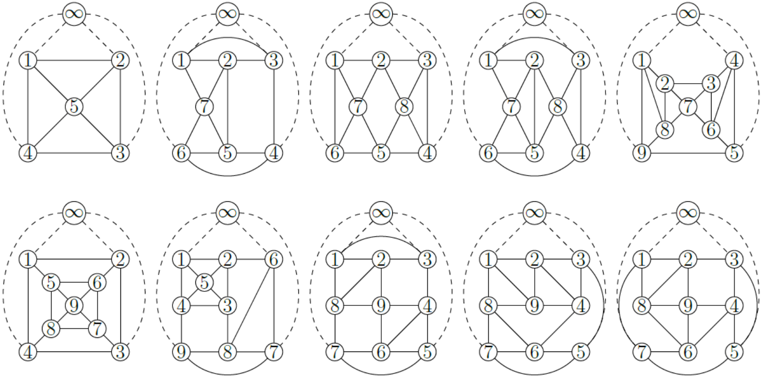
Non-algebraic tangle diagrams can be identified and classified by computing their unique **maximal algebraic subtangle graph**.



In this diagram, $s, s' \in \mathrm{Hom}_{\mathscr{C}_{\mathrm{ASG}(T)}}(G, F)$, with $s = (b_1, b_2, b_3, b_4)$ and $s' = (b_4, b_2, b_1, b_3)$. Hence, $s' = \sigma(s)$, where $\sigma = (1, 3, 4) = (3, 4)(1, 4)$ is a permutation of bigon collapses which commute.

Non-Algebraic Diagrams and Constellations

The MASG of all non-algebraic tangle diagrams with at most 9 crossings can be described using one of the first 10 **constellations**.



List of all possible $k+1$ constellations with $k+1 \leq 10$

Thank You!