Fast Erasure Decoder for a Class of Quantum LDPC Codes

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Contributors

https://arxiv.org/abs/2208.01002

   

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PART 1

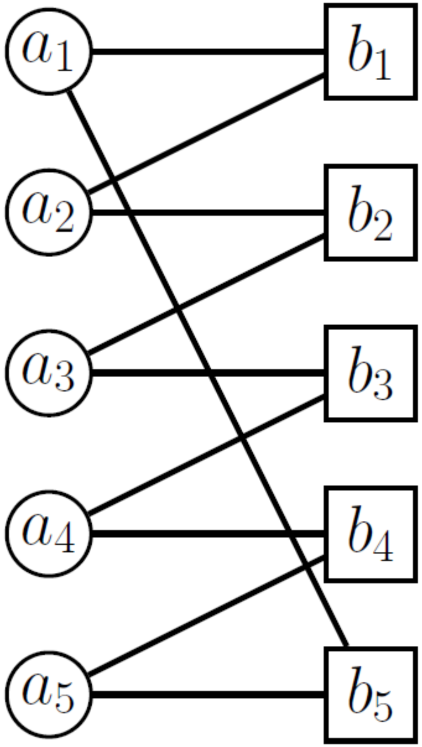
Classical Error Correcting Codes:
A Brief Reminder

Recall: Classical Codes

- **A classical <u>linear code</u> *C* is a vector space over Z_2.**
  - A code of <u>length</u> n is the kernel of an $r \times n$ <u>parity check matrix</u> H.
  - C has <u>dimension</u> k as s subspace of $\mathbf{Z}_2^n$.
- **Vectors x in C are <u>codewords</u>.**
- **C is visualized by its bipartite <u>Tanner graph</u> $T(H)$.**

$$H = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad \ker(H) = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

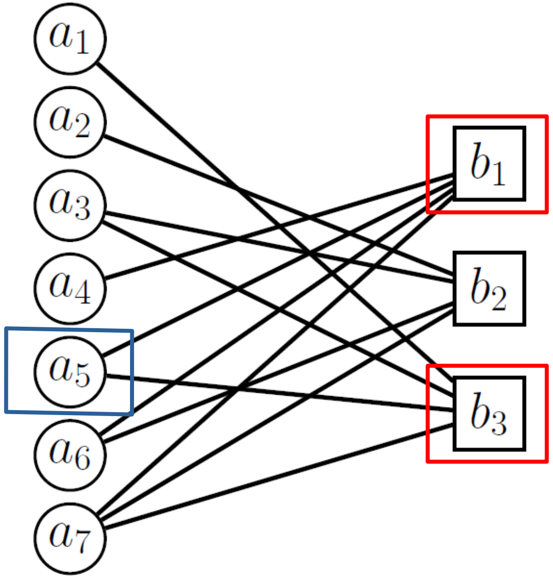
$T(H) =$



The Tanner Graph of a Code

- **The parity check matrix H defines the bipartite Tanner graph $T(H)$.**
  - The columns of H define the bit vertices: $A = \{a_1, a_2, \ldots, a_n\}$.
  - The rows of H define the check vertices: $B = \{b_1, b_2, \ldots, b_r\}$.
  - There exists an edge between a_i and b_j if and only if $H_{i,j} \neq 0$.

$$H = \begin{array}{c c} \begin{array}{ccccccc} & & & & a_5 & & \end{array} & \\ \left[\begin{array}{ccccccc} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{array}\right] & \begin{array}{c} b_1 \\ \\ b_3 \end{array} \end{array}$$



Recall: Classical Error Correction

$$\overset{\text{message}}{\underset{x}{\begin{bmatrix}1\\1\\1\\1\\1\end{bmatrix}}} \Rightarrow \overset{\text{corruption}}{\underset{x+e=y}{\begin{bmatrix}1\\1\\1\\1\\1\end{bmatrix}+\begin{bmatrix}0\\1\\0\\0\\0\end{bmatrix}=\begin{bmatrix}1\\0\\1\\1\\1\end{bmatrix}}} \Rightarrow \overset{\text{syndrome}}{\underset{s=Hy}{\begin{bmatrix}1\\1\\0\\0\\0\end{bmatrix}}}$$

$$\overset{\text{prediction}}{\underset{\hat{e}}{\begin{bmatrix}0\\1\\0\\0\\0\end{bmatrix}}} \Rightarrow \overset{\text{correction}}{\underset{y+\hat{e}}{\begin{bmatrix}1\\0\\1\\1\\1\end{bmatrix}+\begin{bmatrix}0\\1\\0\\0\\0\end{bmatrix}}} \Rightarrow \overset{\text{recovery}}{\underset{x}{\begin{bmatrix}1\\1\\1\\1\\1\end{bmatrix}}}$$

1. Send initial codeword x in C in Z_2^n.
2. Receive corrupted codeword $y = x + e$ in Z_2^n.
3. Make syndrome measurement $s = Hy = He$ in Z_2^r .
4. Decoder predicts an error $\hat{e}$ satisfying $s = H\hat{e}$.
5. Perform error correction $y + \hat{e}$.
6. Recover original codeword if $y + \hat{e} = x$.

Binary Erasure Channel

- **The <u>binary erasure channel</u> erases each bit with probability *p*.**
  - The set of erased bits ε is known.
- **Erasure correction can be achieved using error correction.**
  - Erased bits are assigned random values.

$$\begin{array}{ccccc} \text{message} & & \text{erasure} & & \text{assignment} \\ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} & \Rightarrow & \begin{bmatrix} 1 \\ ? \\ 1 \\ ? \\ ? \end{bmatrix} & \Rightarrow & \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \\ x & & & & y \end{array}$$

PART 2

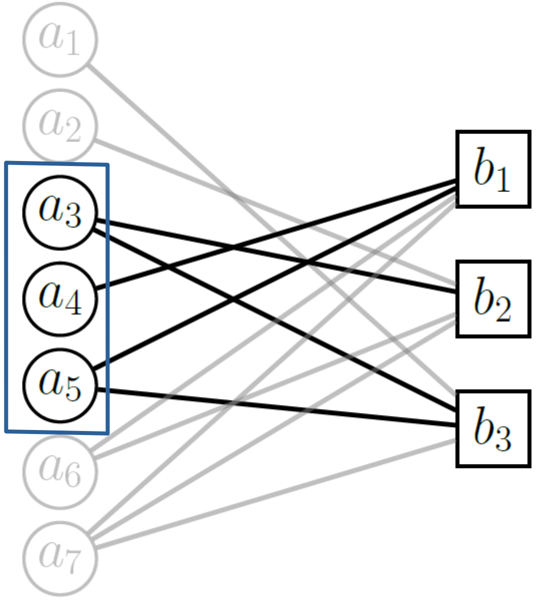
The Peeling Decoder

Erasure-Induced Subgraph of the Tanner Graph

- An erasure **ε** induces a subgraph of the Tanner graph *T(H)*.
  - **Example**: $\varepsilon = \{a_3, a_4, a_5\}$.
- We can use information about **ε** to perform correction.
  - Non-erased bits do not have errors.

$$x' = \begin{bmatrix} 1 \\ 1 \\ ? \\ ? \\ ? \\ 1 \\ 0 \end{bmatrix}$$

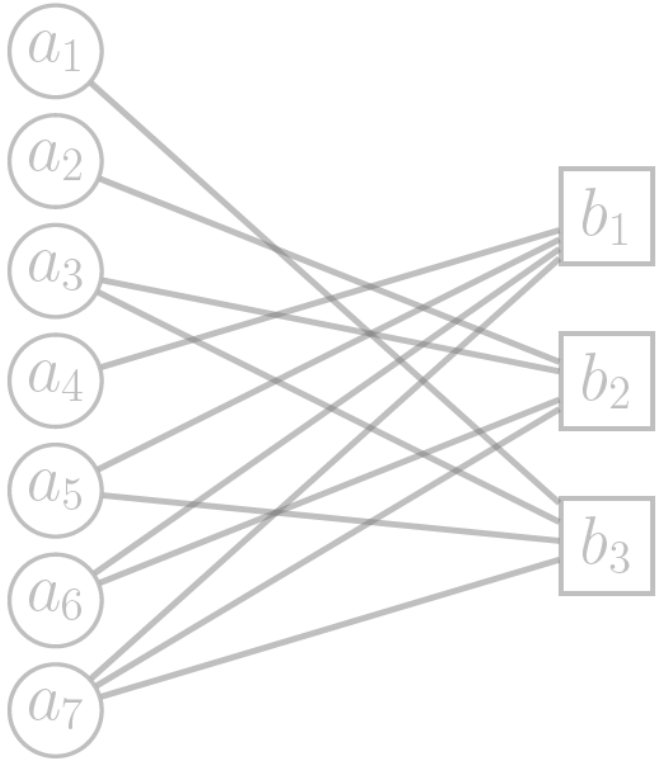
$$H = \begin{array}{c} \begin{array}{ccccccc} & & a_3 & a_4 & a_5 & & \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix} \end{array}$$



Algorithm: Peeling Dangling Checks

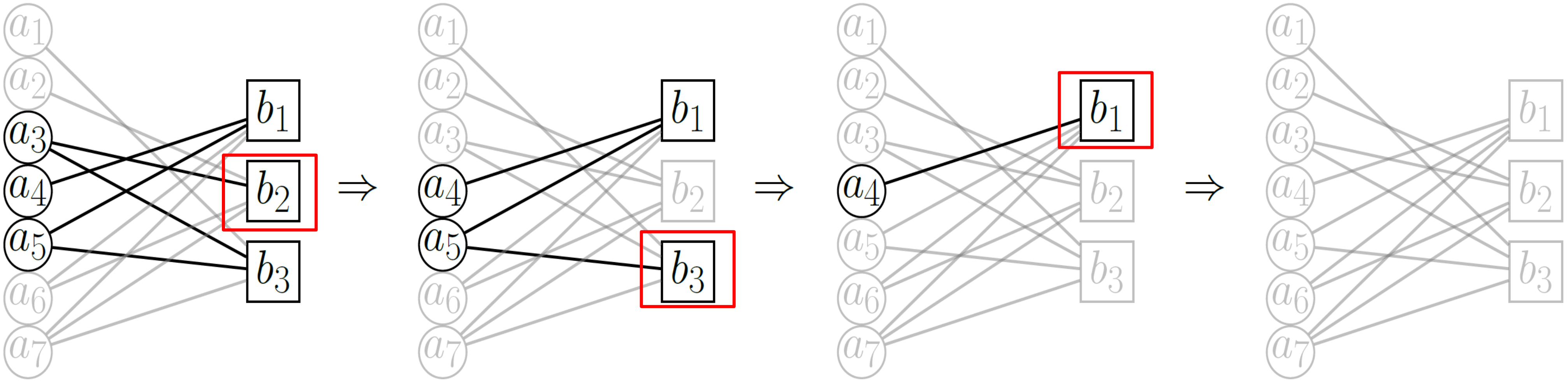
1. Given erasure pattern ε, consider the induced subgraph of *T(H)*.
2. Select a dangling (degree 1) check in this subgraph.
3. Correct the adjacent bit and remove it from ε (shrinking the subgraph).
4. Algorithm terminates when ε is empty (or gets stuck in a stopping set).

The complexity of the peeling decoder is linear in the number of bits.



Peeling Decoder: Full Example of a Decoder Success

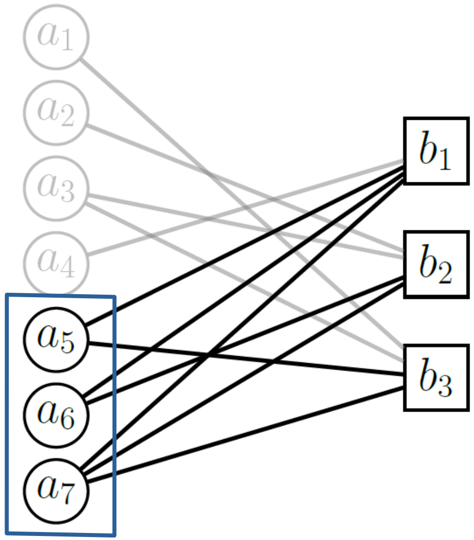
- Decoding success or failure depends <u>only</u> on the erasure-induced subgraph of *T(H)*.
- Success occurs when there exists a sequence of dangling checks that fully “peel” ***ε***.



Stopping Sets for the Peeling Decoder

- An erasure-induced subgraph of *T(H)* with no dangling checks is a <u>stopping set</u> for the peeling decoder (the decoder fails).
- Tanner graphs for sparse codes generally have fewer stopping sets.

$$H = \begin{array}{c} \begin{array}{ccccccc} & & & & a_5 & a_6 & a_7 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix} \end{array}$$



Quantum Code Review: Hypergraph Product Codes

Review: Families of Quantum Codes

- **Recall that a <u>quantum code</u> of length *N* and dimension *K* is a subspace of a Hilbert space.**
  - Vectors are N-qubit states $|\psi\rangle$ in $\mathbf{C}^N$.
  - Errors on a state $|\psi\rangle$ are described by N-qubit Pauli operators in $P_N = \{I,X,Z,Y\}^{\times N}$.
- **<u>Stabilizer Codes</u> are the space of states left fixed by a subgroup of the Pauli group P_N.**
- **<u>CSS Codes</u> are stabilizer codes defined by commuting *N*-qubit *X*- and *Z*-Pauli operators.**
  - X- and Z-Pauli <u>stabilizer generators</u> define the rows of matrices H_X and H_Z (where $H_X H_Z^T = 0$).
  - CSS Z and X error correction is modeled using classical codes $C_X = \ker(H_X)$ and $C_Z = \ker(H_Z)$.
- **<u>Surface Codes</u> are CSS codes defined from the cellulation of a surface.**
- **<u>Hypergraph Product Codes</u> are another type of CSS code.**

Review: Pauli Errors for CSS Codes

- Pauli errors X_i and Z_i in P_N can be mapped onto binary strings e_i in Z_2^N.

$$E = X_1 Z_1 X_2 Z_4 \in P_4 \quad \Leftrightarrow \quad e_X = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad e_Z = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

- Pauli error correction for a CSS code can be modeled as classical error correction using H_X or H_Z (handling X and Z errors separately).
- The peeling decoder algorithm can be directly applied to CSS codes.

Hypergraph Product Codes: Definition

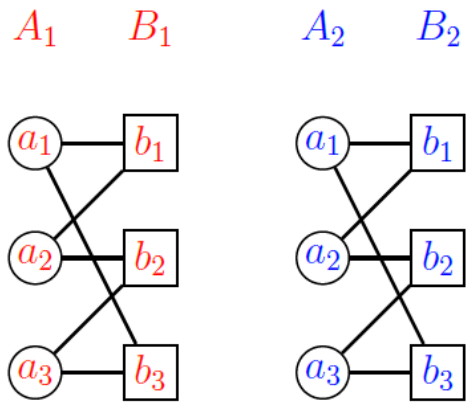
Theorem (Tillich-Zémor):

The Hypergraph Product (HGP) code of two classical codes $C_1 = \ker(H_1)$ and $C_2 = \ker(H_2)$ is the quantum code $C = \mathrm{CSS}(C_X, C_Z)$, where $C_X = \ker(H_X)$ and $C_Z = \ker(H_Z)$ have parity check matrices H_X and H_Z defined from H_1 and H_2 as follows.

$$\begin{aligned} H_X &= \left[\; H_1 \otimes I \;\middle|\; I \otimes H_2^T \;\right] \\ H_Z &= \left[\; I \otimes H_2 \;\middle|\; H_1^T \otimes I \;\right] \end{aligned}$$

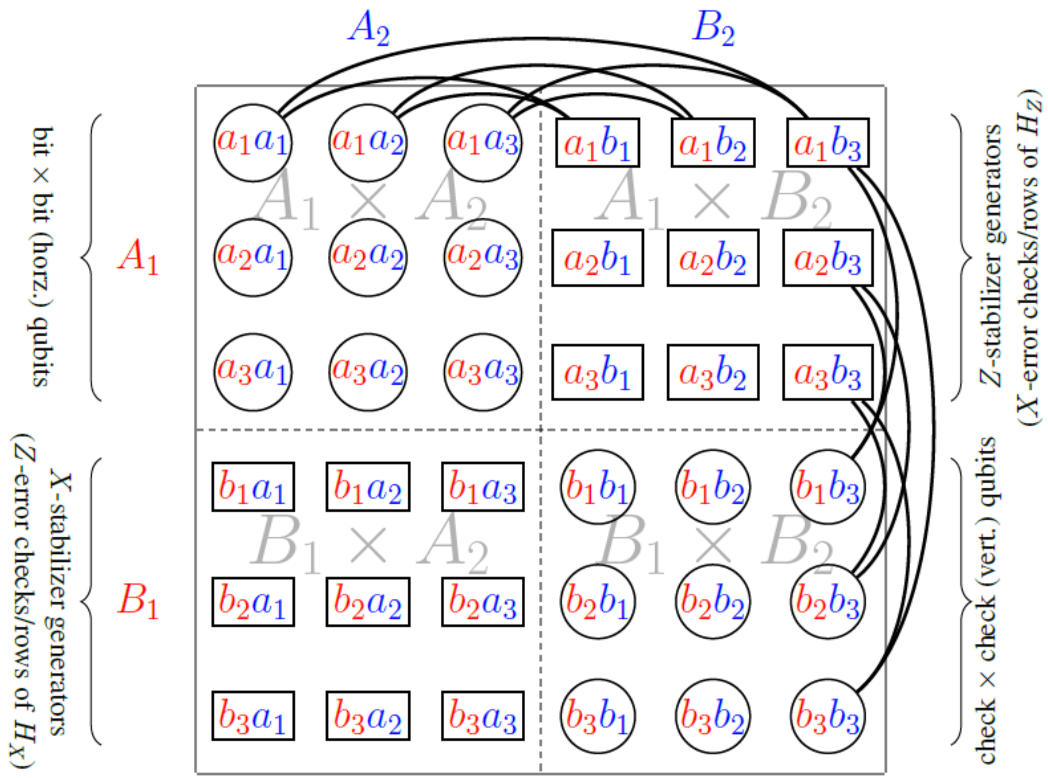
- The matrices have sizes $H_1 = [\, r_1 \times n_1 \,]$, $H_2 = [\, r_2 \times n_2 \,]$, thus $H_X = [\, r_1 n_2 \times (n_1 n_2 + r_1 r_2) \,]$, $H_Z = [\, r_2 n_1 \times (n_1 n_2 + r_1 r_2) \,]$.
- C has length $N = n_1 n_2 + r_1 r_2$ and dimension $K = N - \mathrm{rank}(H_X) - \mathrm{rank}(H_Z)$
- C has minimum distance $\min(d_1, d_2)$, where d_1 and d_2 are the minimum distances of C_1 and C_2.

Hypergraph Product Codes: Tanner Graph Structure



$$T(H_1) \cong T(H_2)$$

$$H_1 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} = H_2$$



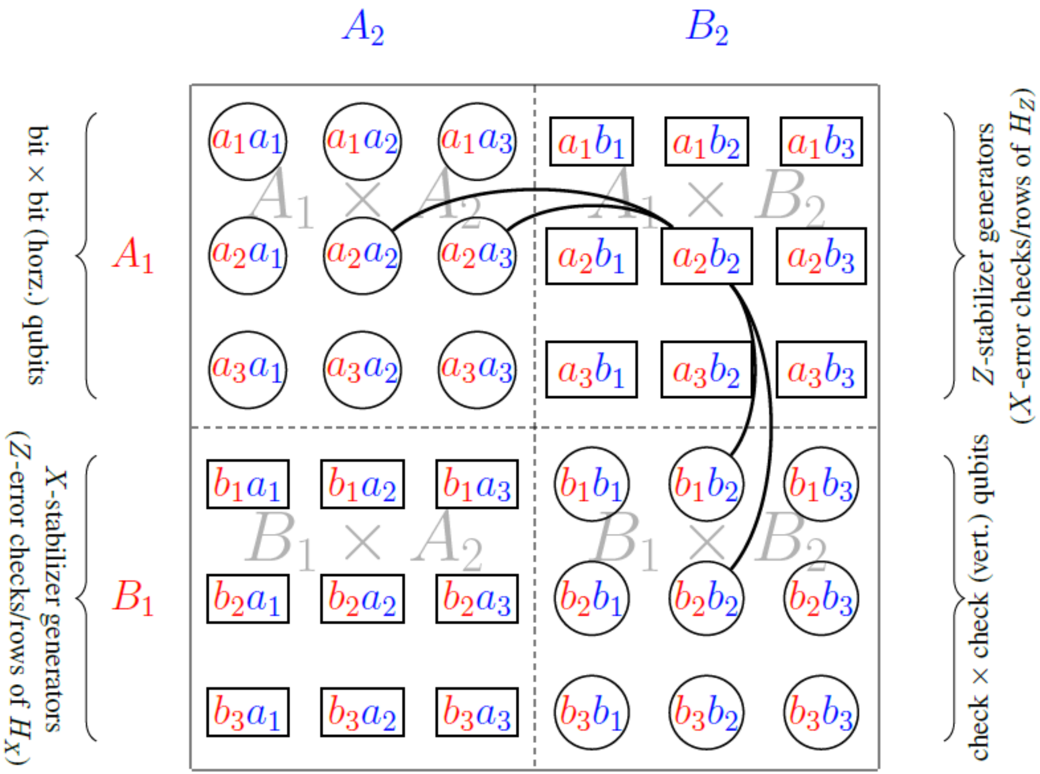
$$H_Z = \begin{bmatrix} 110000000 & 100000100 \\ 011000000 & 010000010 \\ 101000000 & 001000001 \\ 000110000 & 100100000 \\ 000011000 & 010010000 \\ 000101000 & 001001000 \\ 000000110 & 000100100 \\ 000000011 & 000010010 \\ 000000101 & 000001001 \end{bmatrix}$$

$$[\ I \otimes H_2 \mid H_1^T \otimes I\ ]$$

$$H_X = \begin{bmatrix} 100100000 & 101000000 \\ 010010000 & 110000000 \\ 001001000 & 011000000 \\ 000100100 & 000101000 \\ 000010010 & 000110000 \\ 000001001 & 000011000 \\ 100000100 & 000000101 \\ 010000010 & 000000110 \\ 001000001 & 000000011 \end{bmatrix}$$

$$[\ H_1 \otimes I \mid I \otimes H_2^T\ ]$$

Hypergraph Product Codes: Z-type Stabilizer Generators



a_2a_3

a_2a_2 b_1b_2 b_2b_2

$$H_Z = \begin{bmatrix} 110000000 & 100000100 \\ 011000000 & 010000010 \\ 101000000 & 001000001 \\ 000110000 & 100100000 \\ 000011000 & 010010000 \\ 000101000 & 001001000 \\ 000000110 & 000100100 \\ 000000011 & 000010010 \\ 000000101 & 000001001 \end{bmatrix} \quad a_2b_2$$

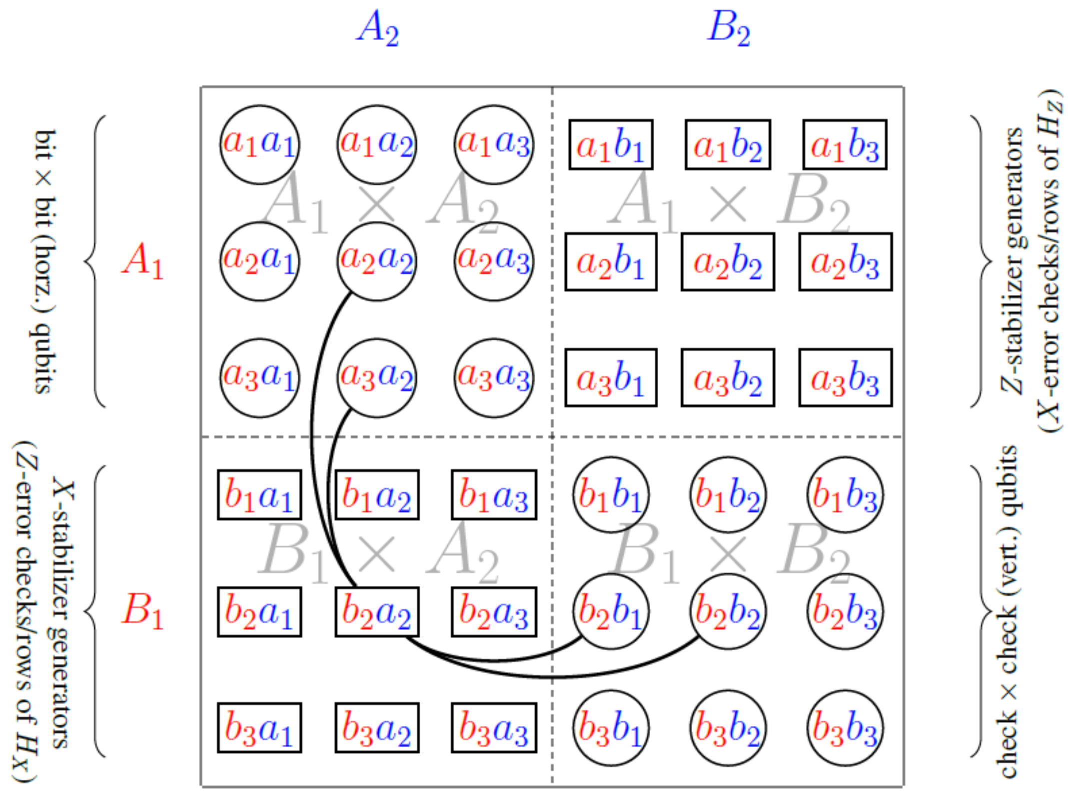
$$\left[\; I \otimes H_2 \;\middle|\; H_1^T \otimes I \;\right]$$

Hypergraph Product Codes: X-type Stabilizer Generators

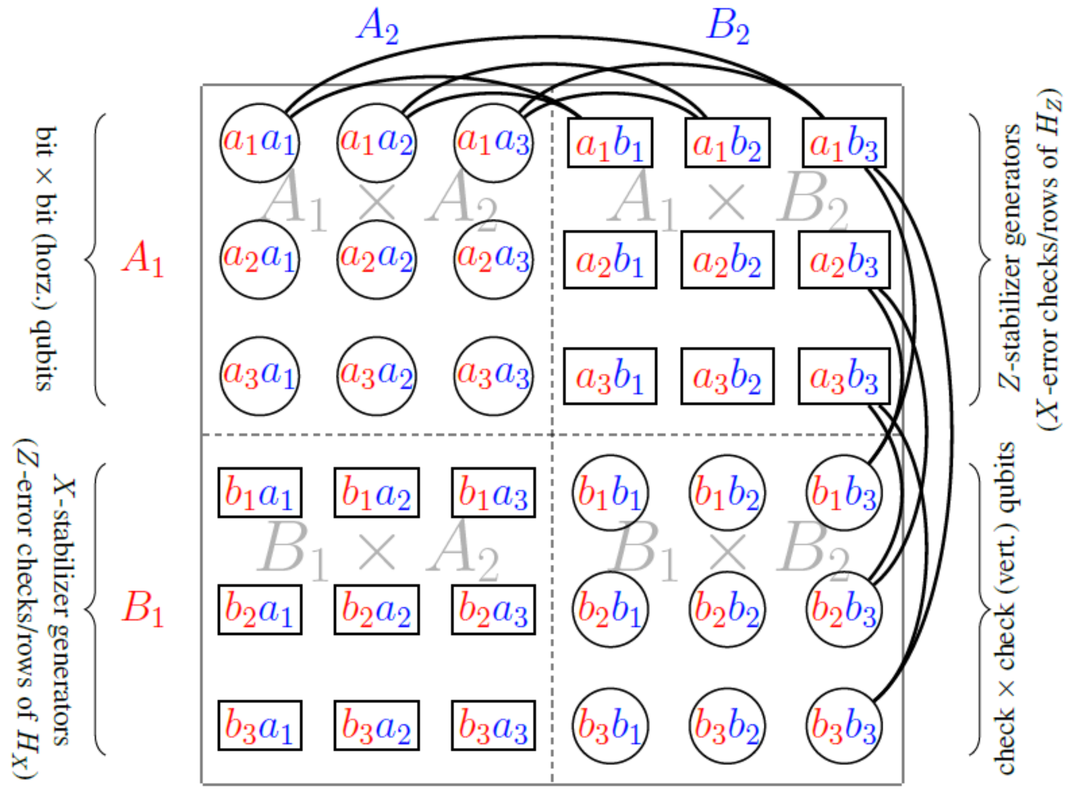
a_2a_2 a_3a_2 b_2b_1 b_2b_2

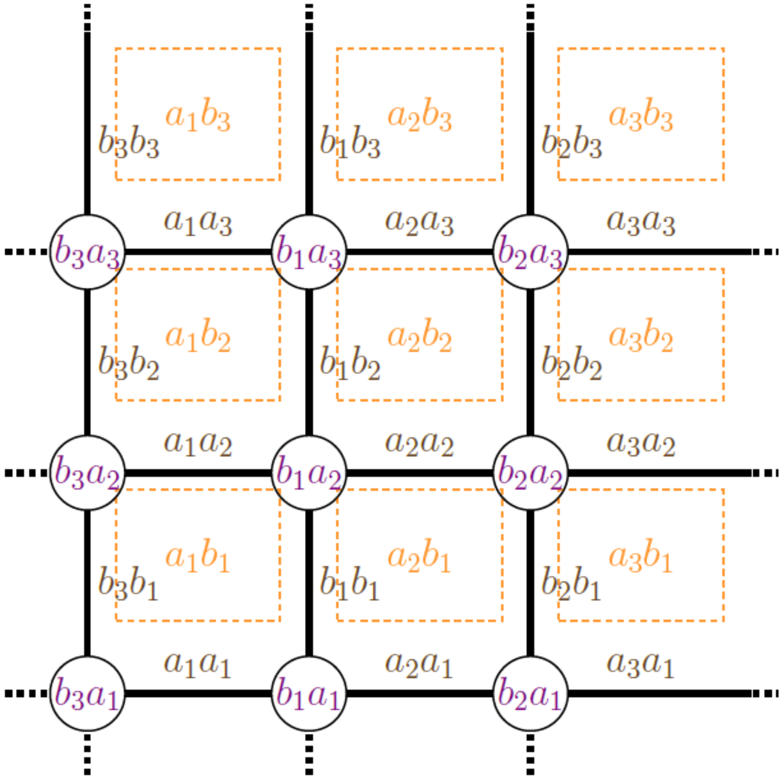
$$H_X = \begin{bmatrix} 100100000 & 101000000 \\ 010010000 & 110000000 \\ 001001000 & 011000000 \\ 000100100 & 000101000 \\ \boxed{000010010 \;\; 000110000} \\ 000001001 & 000011000 \\ 100000100 & 000000101 \\ 010000010 & 000000110 \\ 001000001 & 000000011 \end{bmatrix} \quad b_2a_2$$

$$\left[\; H_1 \otimes I \;\middle|\; I \otimes H_2^T \;\right]$$



Aside: Toric Code HGP Picture versus Lattice Picture





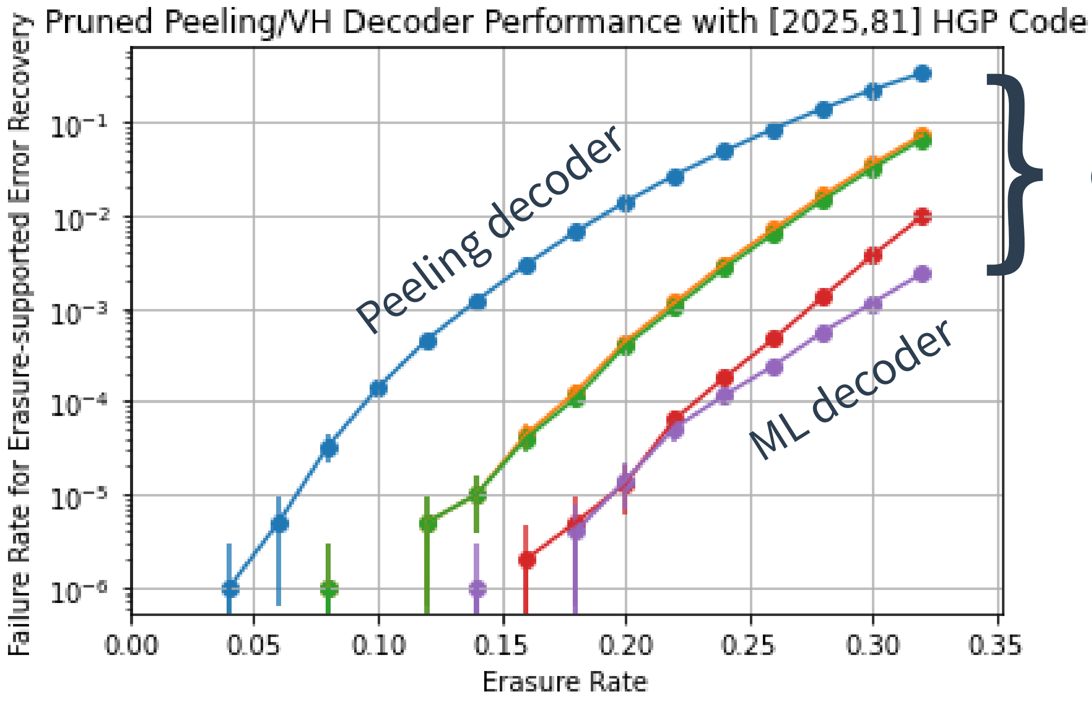
PART 4

Generalized Peeling Decoder for HGP Codes

Peeling Decoder Applied to HGP Codes

- NAIVE IDEA
  - Does the basic peeling decoder perform well when applied to HGP codes?
- PROBLEM
  - In practice, the peeling decoder applied to HGP codes performs poorly.
  - The decoder often fails because of stopping sets unique to HGP codes.
- STRATEGY
  - Modify the decoder to overcome the most common stopping sets.
  - Generalized algorithm combines peeling with additional techniques.

Numerical Preview: Naive Peeling Decoder vs. ML Decoder



Gap between Peeling and ML decoders

CASE 1: Stabilizer Stopping Sets

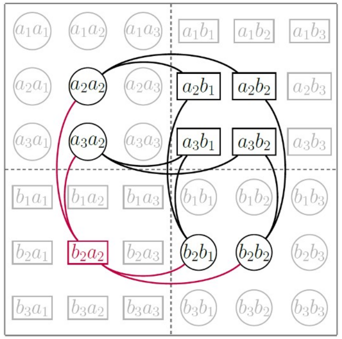
The qubit support of an *X*-type stabilizer is a stopping set for the Tanner graph *T(H_Z)*.

PROOF

- Each *X*-type stabilizer commutes with *Z*-type stabilizer generators by construction $(H_Z H_X^T = 0)$.
- The binary representation of an *X*-stabilizer is a codeword for the classical code $C = \ker(H_Z)$.
- Each row of H_Z (*Z* generator) is adjacent to an even number of qubits in the support of the *X*-stabilizer.
- The subgraph induced by this support contains no degree 1 checks (hence, it is a stopping set).

$$H_X = \begin{bmatrix} 100100000 & 101000000 \\ 010010000 & 110000000 \\ 001001000 & 011000000 \\ 000100100 & 000101000 \\ 000010010 & 000110000 \\ 000001001 & 000011000 \\ 100000100 & 000000101 \\ 010000010 & 000000110 \\ 001000001 & 000000011 \end{bmatrix} e_X \qquad e_X = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ \vdots \end{bmatrix}$$

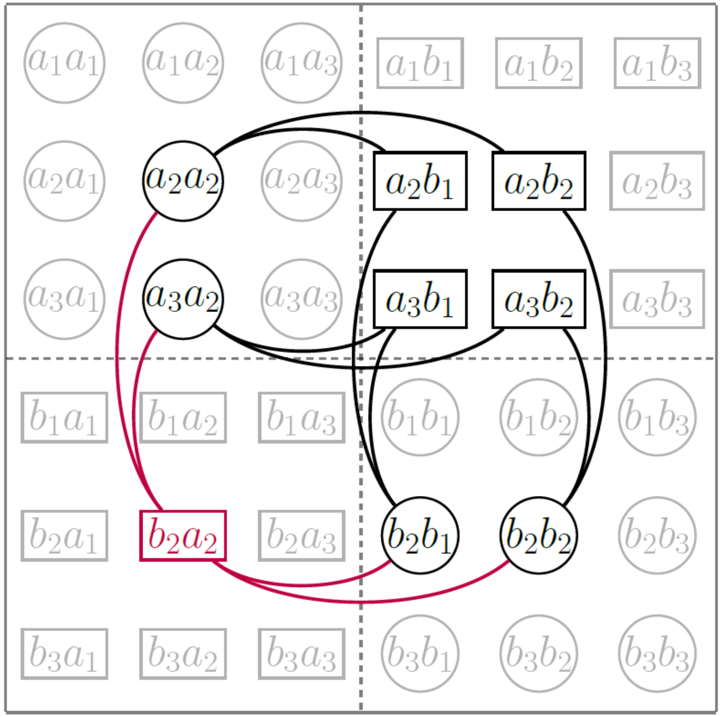
$$H_Z e_X = \begin{bmatrix} 110000000 & 100000100 \\ 011000000 & 010000010 \\ 101000000 & 001000001 \\ 000110000 & 100100000 \\ 000011000 & 010010000 \\ 000101000 & 001001000 \\ 000000110 & 000100100 \\ 000000011 & 000010010 \\ 000000101 & 000001001 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ \vdots \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$



Visualizing Stabilizer Stopping Sets in the Tanner Graph

Column labels (above both matrices): a_2a_2, a_3a_2, b_2b_1, b_2b_2

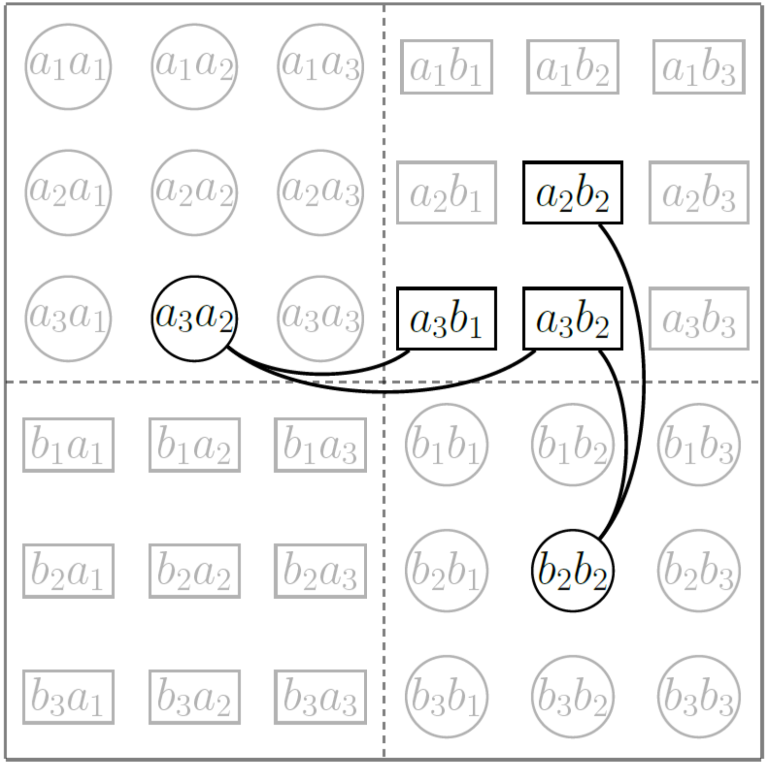
$$H_X = \begin{array}{c|c} \begin{bmatrix} 100100000 & 101000000 \\ 010010000 & 110000000 \\ 001001000 & 011000000 \\ 000100100 & 000101000 \\ \boxed{000010010} & \boxed{000110000} \\ 000001001 & 000011000 \\ 100000100 & 000000101 \\ 010000010 & 000000110 \\ 001000001 & 000000011 \end{bmatrix} & \begin{matrix} \\ \\ \\ \\ b_2a_2 \\ \\ \\ \\ \\ \end{matrix} \end{array}$$



$$H_Z = \begin{array}{c|c} \begin{bmatrix} 110000000 & 100000100 \\ 011000000 & 010000010 \\ 101000000 & 001000001 \\ \boxed{000110000} & \boxed{100100000} \\ \boxed{000011000} & \boxed{010010000} \\ 000101000 & 001001000 \\ \boxed{000000110} & \boxed{000100100} \\ \boxed{000000011} & \boxed{000010010} \\ 000000101 & 000001001 \end{bmatrix} & \begin{matrix} \\ \\ \\ a_2b_1 \\ a_2b_2 \\ \\ a_3b_1 \\ a_3b_2 \\ \\ \end{matrix} \end{array}$$

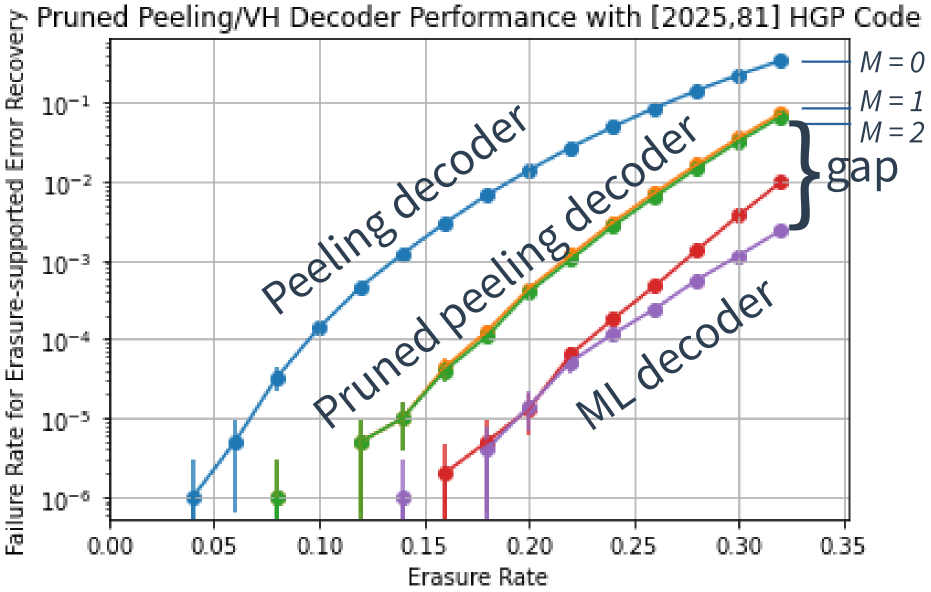
Algorithm: Pruned Peeling Decoder

1. Given erasure pattern ε, apply the standard peeling decoder until stuck.
2. Check whether ε contains the qubit-support S of a stabilizer.
   - If so, then S is a stabilizer stopping set.
3. Break S by removing some qubit from the erasure ε, shrinking the subgraph.
   - This is possible since errors are corrected up to multiplication by a stabilizer.
4. Continue with the peeling decoder.

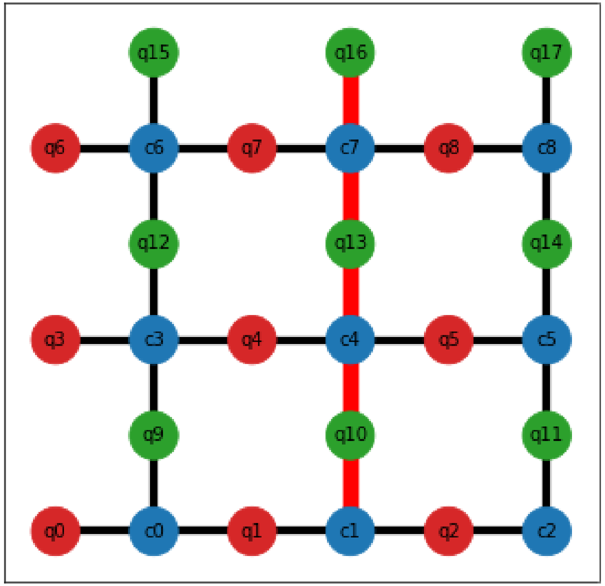


Restrictions on Searching for Stabilizer Stopping Sets

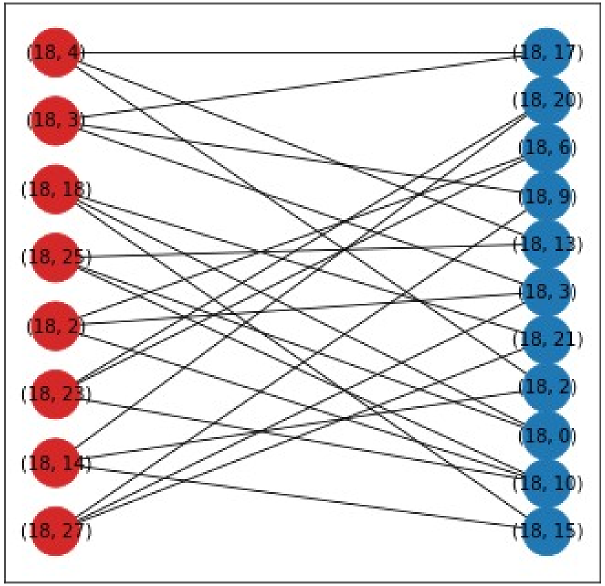
- **Any product of *M* stabilizer generators defines a valid X-stabilizer.**
  - Naive peeling corresponds to $M = 0$.
  - Using only single X-stabilizer generators corresponds to $M = 1$ (the rows of H_X).
  - It is not easy to search for arbitrary products of stabilizer generators with large values of M.
- **Numerically, we see almost no performance increase for large *M*.**
  - The gap is negligible between $M = 1$ and $M = 2$.
  - We only consider up to $M = 2$ in simulations.



Subgraph Induced by Pruned Peeling Decoder Stopping Sets



Subgraph for a stopping set of the 3 × 3 toric code shown on the standard lattice.

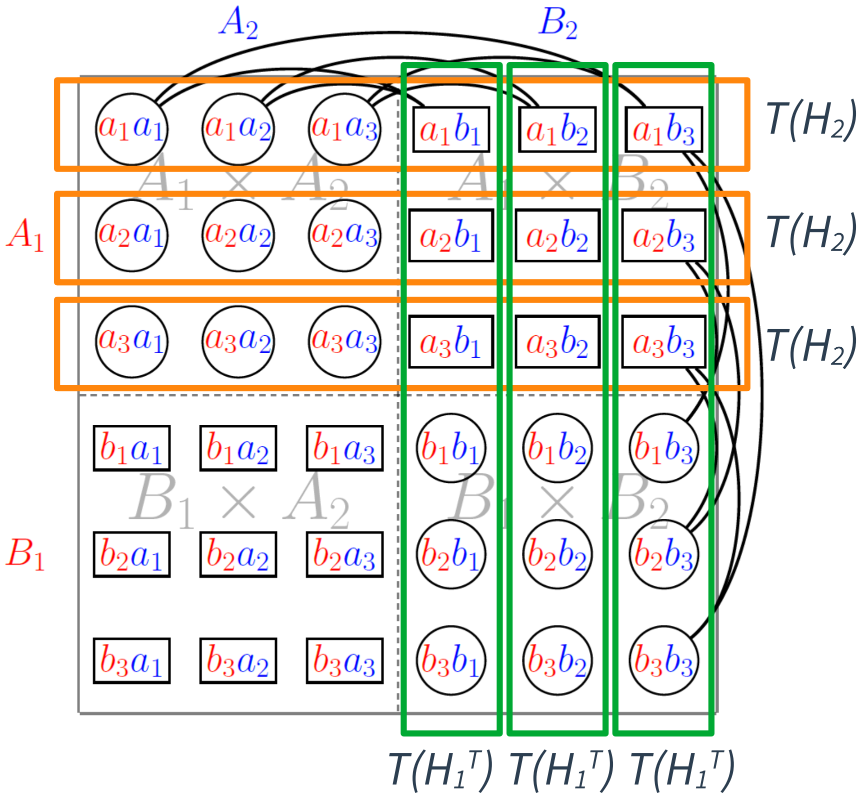


Example of the subgraph induced by a stopping set for a 1600-qubit HGP code.

CASE 2: Classical Stopping Sets

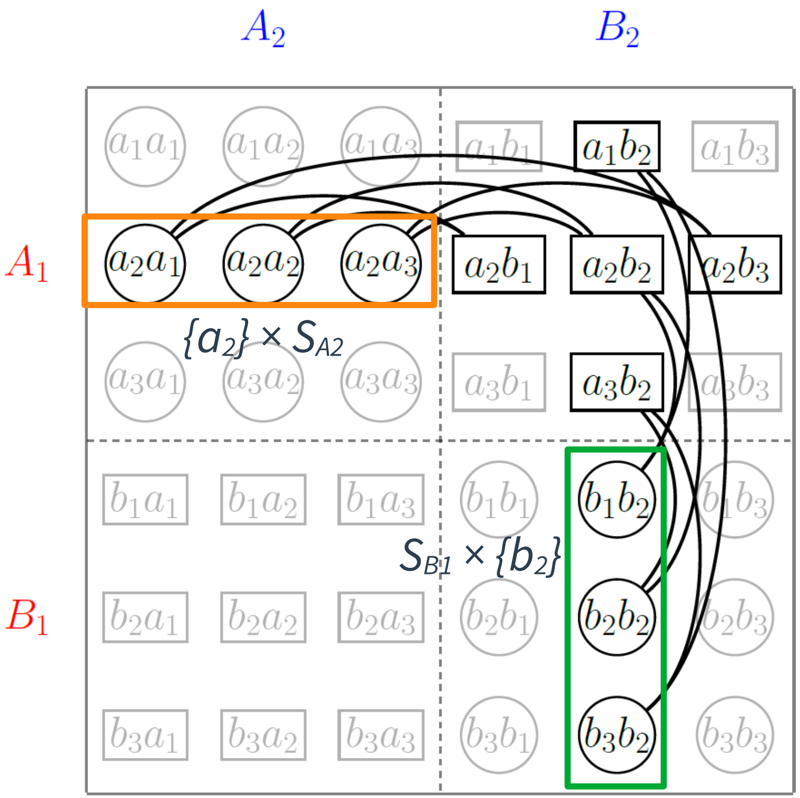
- **The Tanner graph $T(H_Z)$ contains copies of $T(H_2)$ and $T(H_1^T)$ as subgraphs.**
  - Horizontal copies of $T(H_2)$.
  - Vertical copies of $T(H_1^T)$.
- **Stopping sets for $T(H_2)$ and $T(H_1^T)$ lift to stopping sets of $T(H_Z)$ on a row or column.**
- **These are horizontal and vertical <u>classical stopping sets</u> for $T(H_Z)$ in the HGP code.**

$$H_Z = \left[\ I \otimes \boxed{H_2}\ \middle|\ \boxed{H_1^T} \otimes I\ \right]$$



Formalizing Horizontal and Vertical Stopping Sets

- **The HGP Tanner graph $T(H_Z)$ is the product of two bipartite classical Tanner graphs.**
  - $T(H_1) = (A_1 \cup B_1, E_1)$, where A_1 = bits, B_1 = checks.
  - $T(H_2) = (A_2 \cup B_2, E_2)$, where A_2 = bits, B_2 = checks.
- **Classical stopping sets in $T(H_Z)$ can be decomposed into classical components.**
  - Horizontal stopping sets have the form $\{a_i\} \times S_{A2}$ in $A_1 \times A_2$, where S_{A2} is a stopping set of $T(H_2)$.
  - Vertical stopping sets have the form $S_{B1} \times \{b_j\}$ in $B_1 \times B_2$, where S_{B1} is a stopping set of $T(H_1^T)$.



Relative Size and Quantity of Classical Stopping Sets

$$\begin{aligned} H_1 &= [r_1 \times \boxed{n_1}] \\ H_2 &= [r_2 \times \boxed{n_2}] \end{aligned} \quad \Rightarrow \quad \begin{aligned} H_X &= [r_1 n_2 \times \boxed{(n_1 n_2 + r_1 r_2)}] \\ H_Z &= [r_2 n_1 \times \boxed{(n_1 n_2 + r_1 r_2)}] \end{aligned}$$

Classical code lengths n_1 and n_2

HGP code length N

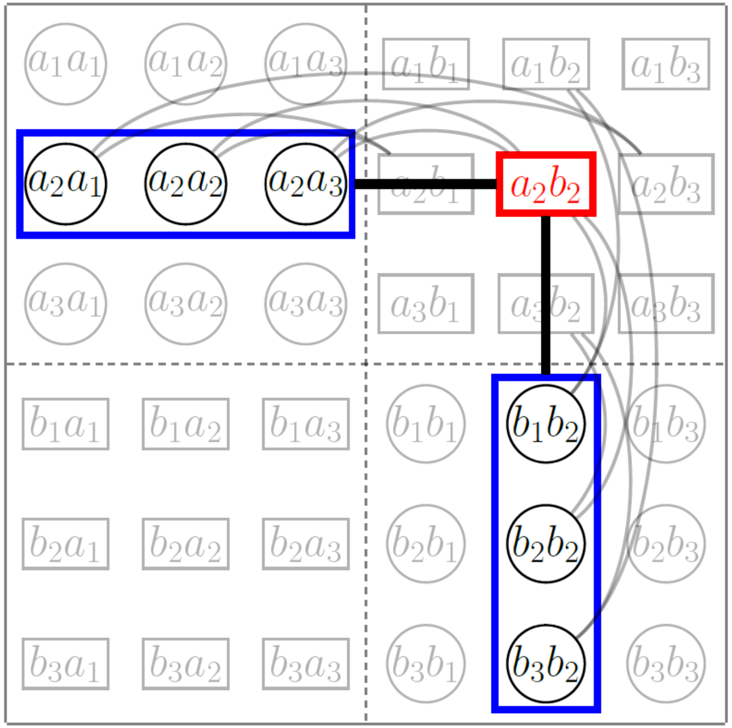
- **The sizes of H_1 and H_2 determine the length N of the HGP code.**
- **Assuming that $n_1 \approx n_2 \approx r_1 \approx r_2$, the classical codes $C_1 = \ker(H_1)$ and $C_2 = \ker(H_2)$ have length $n_1 = O(\sqrt{N}) = n_2$ when compared with the length of the HGP code.**
- **For each classical stopping set of $T(H_2)$ and $T(H_1^T)$, the Tanner graph $T(H_Z)$ contains on the order of $O(\sqrt{N})$ horizontal and vertical stopping sets.**

Further Generalizing the Pruned Peeling Decoder

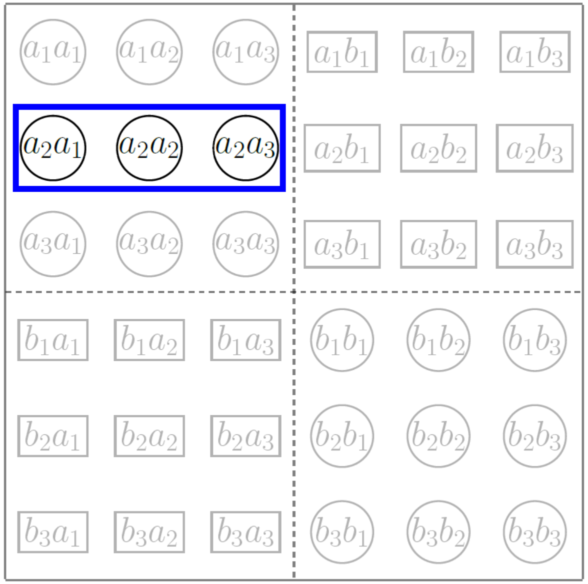
- **OBSERVATION**
  - Numerically, the majority of Pruned Peeling Decoder stopping sets are classical.
- **INTUITION**
  - The maximum likelihood decoder uses cubic complexity Gaussian elimination, which is too slow; but can it be applied efficiently to smaller classical stopping sets?
- **CONSIDERATIONS**
  - If there exist multiple classical stopping sets, how do they interact with each other?
  - Are classical stopping set solutions always consistent with the HGP solution?
  - In combination with peeling, can these stopping sets always be eliminated?

The Vertical-Horizontal (VH) Graph

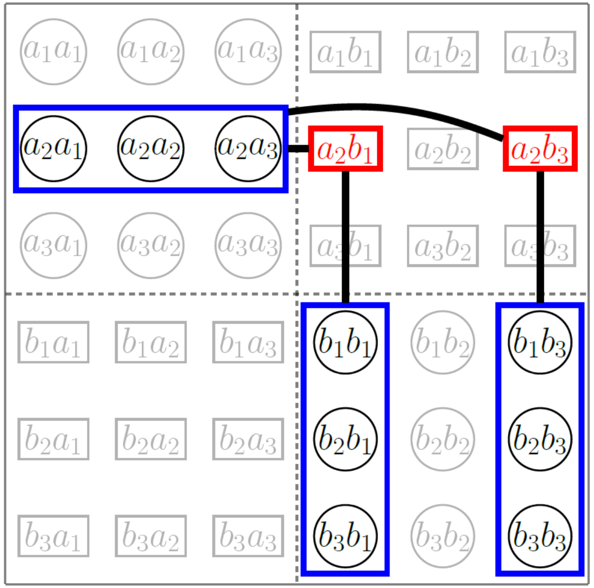
- **Given an erasure pattern ε, define the <u>vertical-horizontal graph</u> as follows.**
  - Vertices are <u>clusters</u> of erased qubits in the same connected component and row/column of $T(H_Z)$.
  - There exists an edge between clusters if there exists a check in $T(H_Z)$ adjacent to a qubit in each.
- **The VH graph is closely related to the erasure-induced subgraph of $T(H_Z)$.**
  - Any two clusters share at most one check (edge).
  - There does not exist an edge between two clusters of the same type (horizontal or vertical).
  - In other words, the VH graph is bipartite.



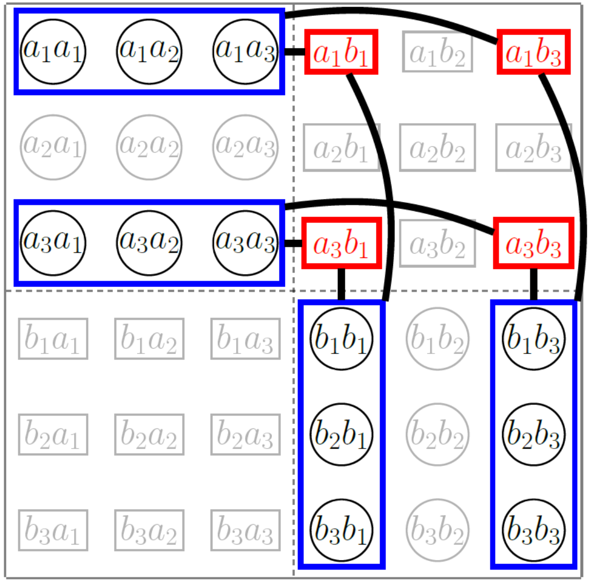
Types of Cluster Configurations in the VH Graph



Isolated Cluster



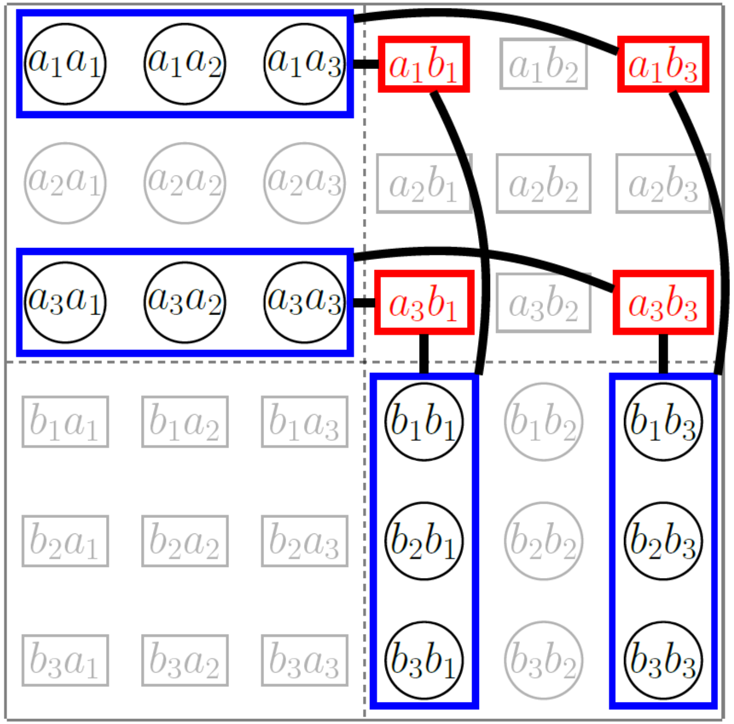
Cluster Tree



Cluster Cycle

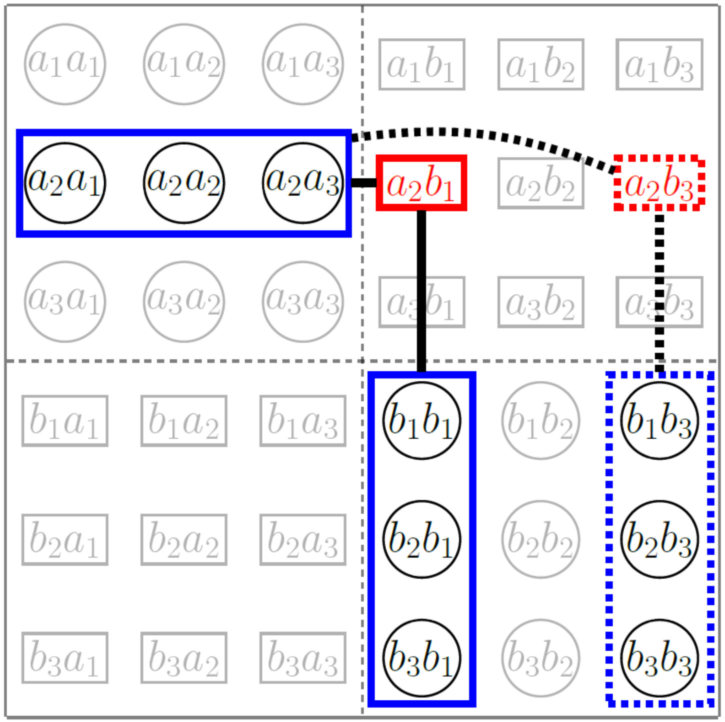
Algorithm: Vertical-Horizontal (VH) Decoder

1. Given erasure pattern ε, apply the pruned-peeling decoder until stuck in a stopping set.
2. Compute the VH-graph of ε.
3. If there exist <u>isolated clusters</u>, solve cluster using Gaussian elimination, then lift solution.
4. If there exist <u>dangling clusters</u>, search for a solution in sequence*, then continue peeling.
   - The order and steps depend on the clusters.
5. If there exist no remaining dangling clusters nor checks, this is a VH decoder stopping set.
   - For example, a cycle of clusters in the VH graph.



Peeling Dangling Clusters in Sequence

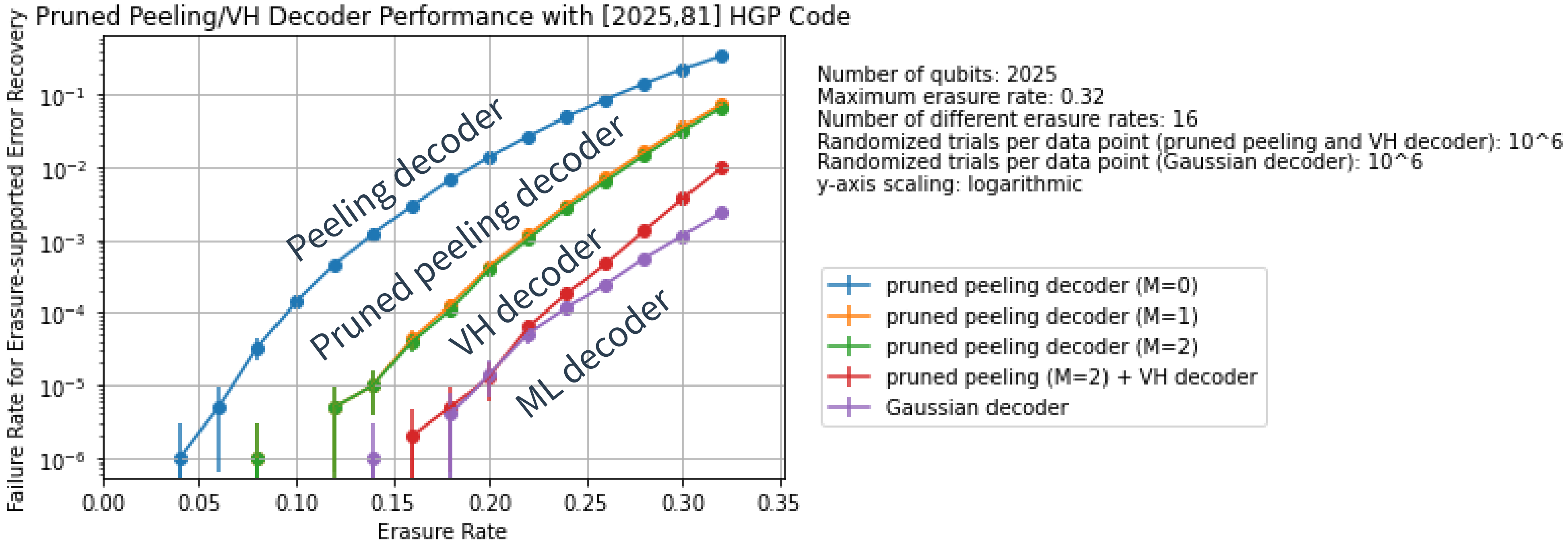
- An edge between two clusters in the VH graph defines a <u>connecting check</u> in *T(HZ)*.
- There are two possibilities for dangling clusters.
  - A connecting check is <u>free</u> if it is the (weight 1) syndrome of a vector in this dangling cluster.
  - Otherwise, the connecting check is <u>frozen</u>.
- <u>Frozen dangling clusters</u> can be solved like isolated clusters and removed from the graph.
  - Solutions have the same contribution to this check.
- <u>Free dangling clusters</u> can be removed from the VH graph and solved after the other clusters.
  - A cluster solution exists independent of this check.



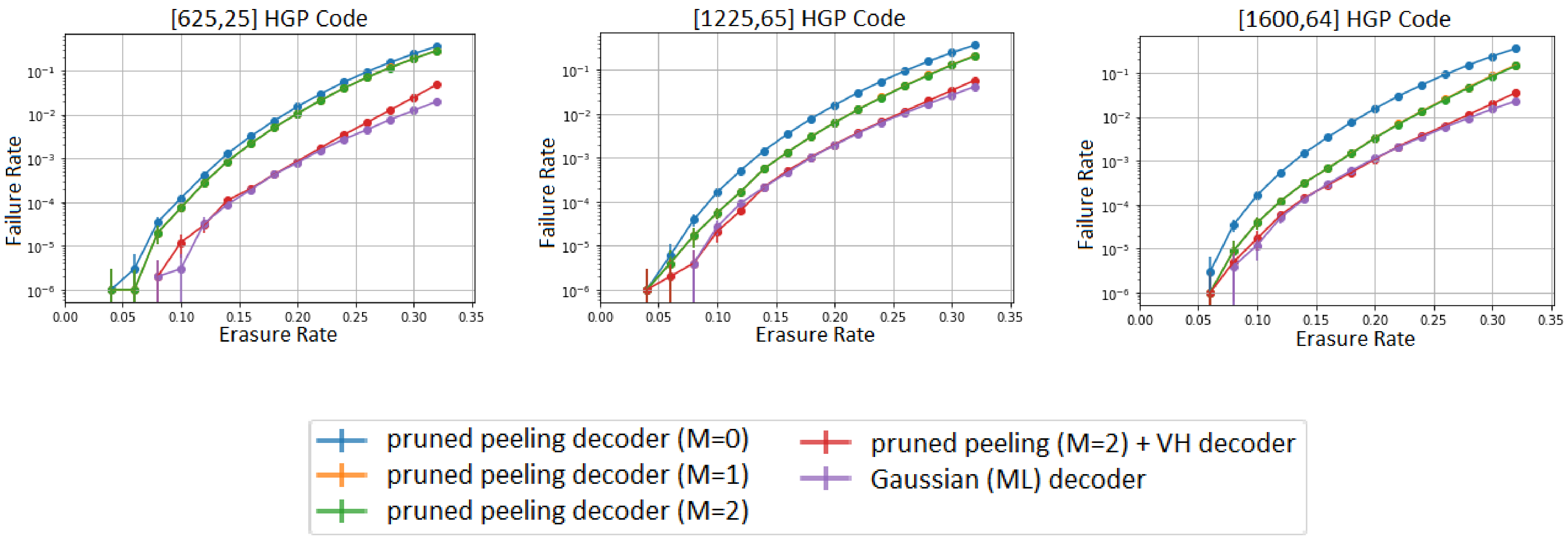
PART 5

Performance of the Pruned Peeling and VH Decoders

Comparison of Performance with Gaussian (ML) Decoder



Performance Comparisons for Other Examples of HGP Codes



Computational Complexity of the Combined Decoder

The computational complexity of combined pruned peeling and VH decoders is dominated by Gaussian elimination applied to clusters.

- Clusters in the VH graph have size $O(\sqrt{N})$, where N is the HGP code length.
- On a single cluster, cubic-complexity Gaussian decoder contributes $O(N^{1.5})$.
- The number of possible clusters grows as $O(\sqrt{N})$.
- Across all clusters, the VH-decoder has complexity $O(N^2)$.
- With a probabilistic implementation of the Gaussian decoder, this can be further reduced to $O(N^{1.5})$ in total.

Thank you!