Tabulation and Classification of 2-String Tangles

Nicholas Connolly

November 13, 2018

"Thus in my mind the most important missing infrastructure project in knot theory is the tabulation of tangles to as high a crossing number as practical."

- Prof. Dror Bar-Natan, University of Toronto,
from a talk given at the University of Iowa in 2016

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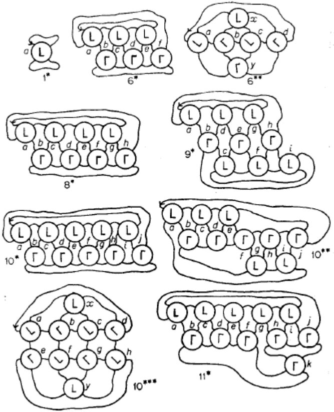
Connection to Knot Theory: Conway's Tabulation[1]

- Conway introduced tangles as a way to tabulate knots.
- Tangles are regarded as portions of knot diagrams.
  - planar polyhedra describe diagrams
  - vertices represent tangles
- All knots up to 11 crossings can be described this way.
  - basic polyhedra (right)
  - algebraic tangle vertices
- Improved on by Caudron.



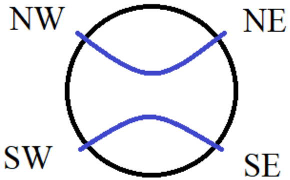
[1] J. H. Conway. An enumeration of knots and lints, and some of their algebraic properties. In *Computational Problems in Abstract Algebra (Proc. Conf., Oxford 1967)*, pages 329-358. Pergamon, Oxford, 1970

Definition

Definition

An **n-string tangle** consists of a 3-dimensional ball with a fixed boundary which has *n* strings properly embedded inside.

- Endpoints are fixed on the surface of the ball.
- For 2-string tangles, label these NW, NE, SE, SW.



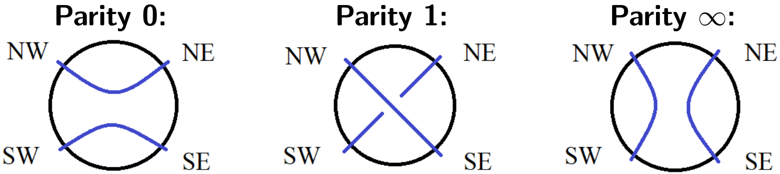
Tangle Diagrams and Parity

Definition

A **tangle diagram** is a 2-dimensional projection of a tangle.

- Crossings distinguish over-strand and under-strand.
- Endpoints are identified by NW, NE, SE, SW (2-string).

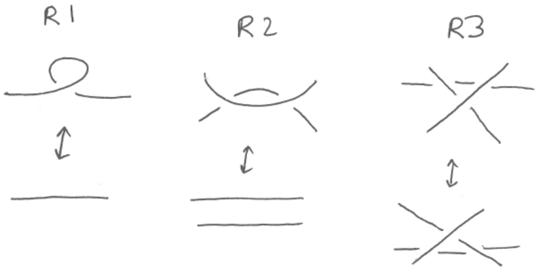
A 2-string tangle has three possible **parities**.



Equivalent Tangles and Reidemeister Moves

Definition

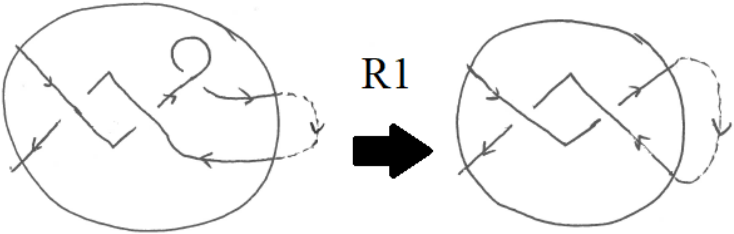
Two tangles are said to be **equivalent** if there exists an ambient isotopy between them which leaves the boundary of the ball fixed.



Two tangle diagrams are equivalent if and only if they are related by a sequence of **Reidemeister moves**.

Equivalent Tangles and Reidemeister Moves

Example:



These are equivalent tangle diagrams related by an R1 move.

Tangle Closure

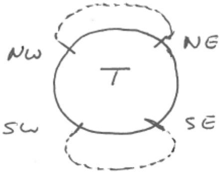
Definition

The **closure** of a 2-string tangle joins the endpoints:

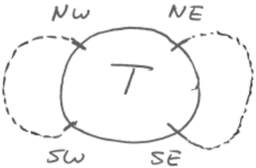
- **numerator:** join NW-NE and SW-SE.
- **denominator:** join NW-SW and NE-SE.

The closure forms a knot or link.

Numerator Closure:



Denominator Closure:



Tangle Orientation

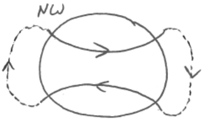
Definition

An **orientation** on a tangle is a choice of direction for each string.

Orientation convention for 2-string tangles:

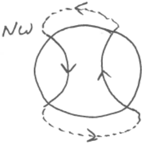
- Travel inward from NW endpoint.
- Use denominator closure for parity 0 or 1.
- Use numerator closure for parity ∞.

Parity 0:



Parity 1:

Parity ∞:



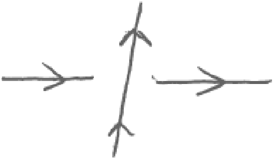
Crossing Sign

Definition

The **sign** of an oriented crossing is determined by the direction of the overstrand and understrand.

Positive Crossing:

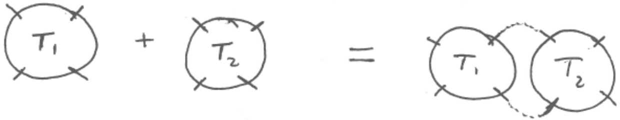
Negative Crossing:

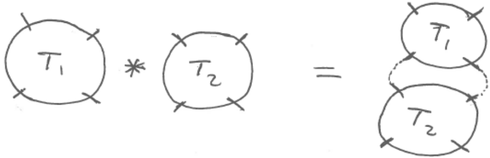


Sum and Product of 2-String Tangles

Definition

A **sum** of tangles $T_1 + T_2$ joins the tangles horizontally as shown.
A **product** of tangles $T_1 * T_2$ joins the tangles vertically as shown.





Tangle Diagram Notations

Dowker Code:

$$-4 \mid -6 \quad -2$$

Gauss Code:

$$|)a1+b2+(|)a3+b1+a2+b3+(|$$

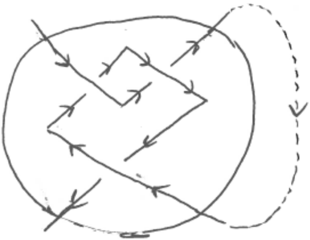
Planar Diagram Code:

$$\begin{array}{ccccc} + & b2 & a3 & d3 & c2 \\ + & b3 & a1 & d1 & c3 \\ + & b1 & a2 & d2 & c1 \end{array}$$

Coloring Matrix:

$$\begin{bmatrix} 1 & -2 & 0 & 1 & 0 \\ -2 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & -2 & 1 \end{bmatrix}$$

Example:



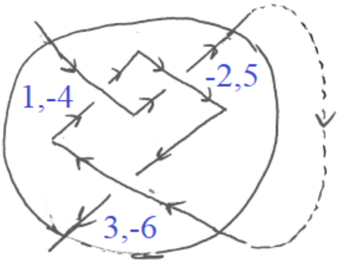
Dowker Code: Algorithm

Algorithm:

1. Start at NW corner.
2. Label crossings with subsequent integers.
3. Each crossing has two labels (even and odd).
4. Even understrand labels are set negative.
5. Order pairs by odd label.
6. Place bar between strings.
7. Code is even sequence.

Example:

$$\begin{array}{ccc} 1 & 3 & 5 \\ -\mathbf{4} & -\mathbf{6} & -\mathbf{2} \end{array}$$



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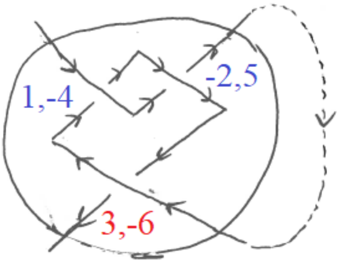


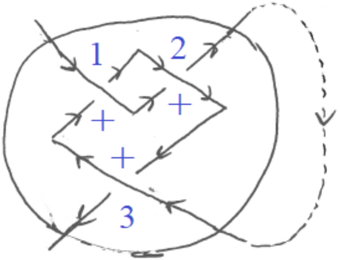
Gauss Code: Algorithm

Algorithm:

1. Start at NW corner.
2. Index crossings with subsequent integers.
3. For each crossing **encounter**, record:
   - above/below (a or b)
   - crossing index
   - crossing sign
4. List crossing information.
5. Place bar between strings.

Example:

$$|)\underbrace{\overbrace{a1+}^{\text{crossing}}\ b2+}_{\text{string 1}}(|)\underbrace{a3+b1+a2+b3+}_{\text{string 2}}(|$$



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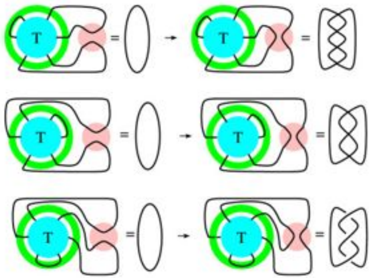
Project Forebears and Contributors

Contributors:

- Isabel K. Darcy
- Jeff Chang
- Nathan Druivenga
- Colin McKinney
- Ram K. Medikonduri
- Stacy Mills
- Junalyn Navarra-Madsen
- Arun Ponnusamy
- Jesse Sweet
- Travis Thompson

Coloring the Mu Transpososome (2006):[2]
Applied tangle analysis to study proteins binding DNA segments.



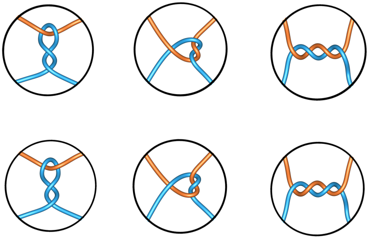
[2] Isabel K Darcy, Jeff Chang, Nathan Druivenga, Colin McKinney, Ram K Medikonduri, Stacy Mills, Junalyn Navarra-Madsen, Arun Ponnusamy, Jesse Sweet, and Travis Thompson. Coloring the mu transpososome. *BMC bioinformatics*, 7(1):435, 2006.

Project Forebears and Contributors

Further Contributors:

- Melanie DeVries
- Hyeyoung Moon
- Rob Scharein
- Danielle Washburn
- Guanyu Wang
- Fengfeng Xia
- Timothy McRoy
- Timothy Gaul
- Matthew Bellmore
- Christine Caples
- Nicholas Connolly*

The Current Project:
Expanded original project from examining 3-string tangles related to DNA to exhaustive tabulation of 2-string tangles.



The Current Project[3]

Goal: Tabulate all 2-string tangles up to specified crossing number.

Method: Computationally generate possible Dowker codes.

Difficulties:

- Parity ambiguity
- Redundant codes
- Non-realizable Dowker codes
- Computational complexity

[3] Contributors: Isabel Darcy, Jeff Chang, Nathan Druivenga, Colin McKinney, Ram Medikonduri, Stacy Mills, Junalyn Navarra-Madsen, Arun Ponnusamy, Jesse Sweet, Travis Thompson, Melanie DeVries, Hyeyoung Moon, Rob Scharein, Danielle Washburn, Guanyu Wang, Fengfeng Xia, Timothy McRoy, Timothy Gaul, Matthew Bellmore, Christine Caples, Nicholas Connolly

Possible Dowker Codes

Dowker Code Components:

1. sequence of even integers (one per crossing)
2. overstrand/understrand (sign of integer)
3. bar placement (before odd label of next strand first crossing)

For a 2-string tangle with n crossings, there are

$$\underbrace{(n!)}_{1}\underbrace{(2^n)}_{2}\underbrace{(n+1)}_{3}$$

possible Dowker codes.

Possible Dowker Codes: Example

For a 2-string tangle with 2 crossings, there are

$$(2!)(2^2)(2+1) = 24$$

possible Dowker codes.

$$\begin{array}{cc|} 2 & 4 \\ -2 & 4 \\ 2 & -4 \\ -2 & -4 \end{array} \qquad \begin{array}{c|c} 2 & 4 \\ -2 & 4 \\ 2 & -4 \\ -2 & -4 \end{array} \qquad \begin{array}{|cc} 2 & 4 \\ -2 & 4 \\ 2 & -4 \\ -2 & -4 \end{array}$$

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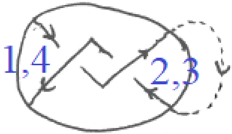
Dowker Difficulties: Parity Ambiguity

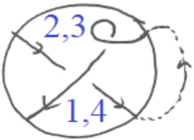
Difficulty: Which tangle matches each Dowker code?

Example: $4|2$

Parity 0:



Parity 1:



Parity ∞:

Solution: Only consider parity 0 tangles.

- Parity ∞ is equivalent to parity 0 by rotation.
- Adding a crossing to a parity 1 tangle makes it parity 0.

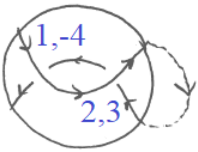
Dowker Difficulties: Redundant Codes: Reducible

Difficulty: Arbitrary tangles may be reducible.

Example: $|2\ 4$

removable R1 moves

Example: $-4|2$



removable R2 move

"Solution:" Only consider minimal tangle diagrams.

Dowker Difficulties: Redundant Codes: Non-reducible

Difficulty: Diagrams related by R3 moves are equivalent.

Example:

$$\begin{array}{ccccc|cc} 1 & 3 & 5 & 7 & 9 & 11 & 13 \\ 14 & 8 & -12 & 2 & 4 & 10 & -6 \end{array}$$

Example:

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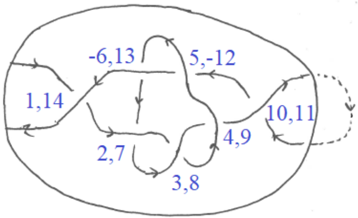
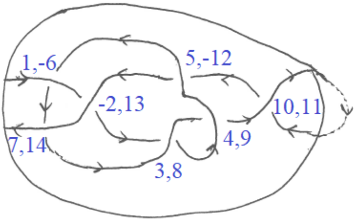




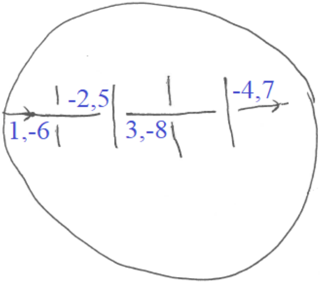
“Solution:” Fix an internal choice for preferred diagrams.

Dowker Difficulties: Realizability

Difficulty: Some codes cannot be realized as tangle diagrams.

Example:

$$\begin{array}{cc|cc} 1 & 3 & 5 & 7 \\ -6 & -8 & -2 & -4 \end{array}$$



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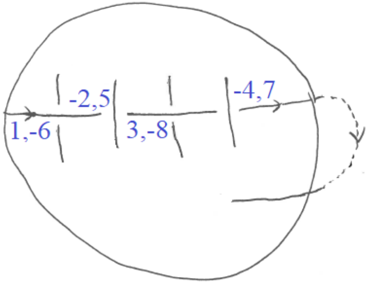

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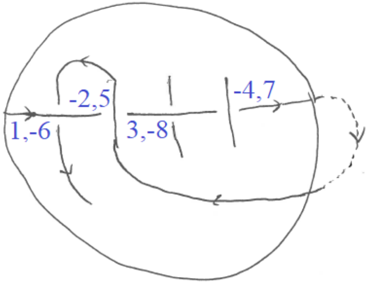

"Solution:" Ignore these Dowker codes.

Possible Dowker Codes: Example

For a 2-string tangle with 2 crossings, there are

$$(2!)(2^2)(2+1) = 24$$

possible Dowker codes. Only two are good.

$$\begin{array}{cc|} 2 & 4 \\ -2 & 4 \\ 2 & -4 \\ -2 & -4 \end{array} \qquad \begin{array}{c|c} 2 & 4 \\ -2 & 4 \\ 2 & -4 \\ -2 & -4 \end{array} \qquad \begin{array}{|cc} 2 & 4 \\ -2 & 4 \\ 2 & -4 \\ -2 & -4 \end{array}$$

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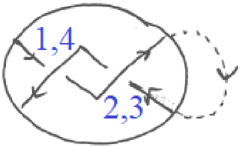
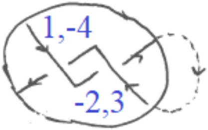
Possible Dowker Codes: Example: Good Codes

$4|2$



$-4|-2$



These two Dowker codes are good because

- they can be realized with a diagram,
- this diagram is reduced,
- this diagram has parity 0.

The Current State of this Project

Accomplishments[4]:

- Dowker code generation
- Diagram realizability
- Notation conversions
- Elimination of many redundancies
- Invariant computations
- Consolidation of existing results in database*

[4] Contributors: Isabel Darcy, Jeff Chang, Nathan Druivenga, Colin McKinney, Ram Medikonduri, Stacy Mills, Junalyn Navarra-Madsen, Arun Ponnusamy, Jesse Sweet, Travis Thompson, Melanie DeVries, Hyeyoung Moon, Rob Scharein, Danielle Washburn, Guanyu Wang, Fengfeng Xia, Timothy McRoy, Timothy Gaul, Matthew Bellmore, Christine Caples, Nicholas Connolly*

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- Notation conversions
- Elimination of many redundancies
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- Consolidation of existing results in database*

Goals:

- Eliminate all redundancies
- Compare with existing tangle lists (classify)
  - rational*
  - Montesinos
  - algebraic
- Compute more invariants
- Generalize notations
- Extend to n-strings

[4] Contributors: Isabel Darcy, Jeff Chang, Nathan Druivenga, Colin McKinney, Ram Medikonduri, Stacy Mills, Junalyn Navarra-Madsen, Arun Ponnusamy, Jesse Sweet, Travis Thompson, Melanie DeVries, Hyeyoung Moon, Rob Scharein, Danielle Washburn, Guanyu Wang, Fengfeng Xia, Timothy McRoy, Timothy Gaul, Matthew Bellmore, Christine Caples, Nicholas Connolly*

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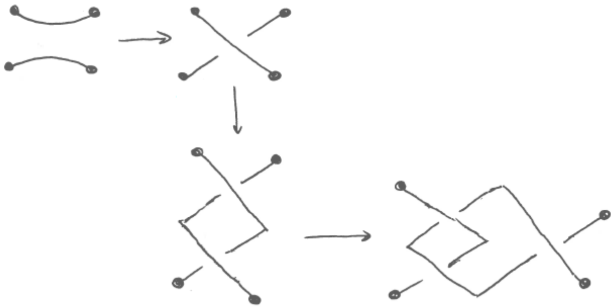
Rational Tangle Intuitive Definition: Twisting Strands

Definition

A tangle T is **rational** if it can be obtained by performing a finite sequence of vertical and horizontal twists on two strands.



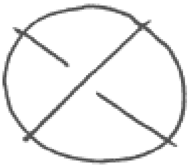
Twisting Tangle Endpoints

Definition

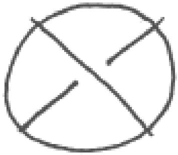
Twisting the endpoints of a tangle is equivalent to adding to or multiplying by a one crossing tangle.

- Twists can be positive [1] or negative [-1].
- Any two adjacent endpoints (N,E,S,W) can be twisted.

Positive Twist: [1]

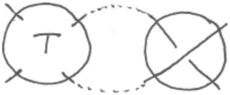


Negative Twist: [-1]

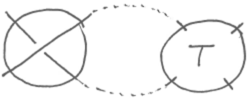


Twisting Tangle Endpoints: Example

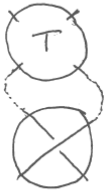
Right Twist: $T + [1]$



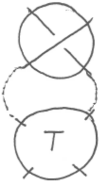
Left Twist: $[1] + T$



Bottom Twist: $T * [1]$



Top Twist: $[1] * T$



Applying Multiple Twists

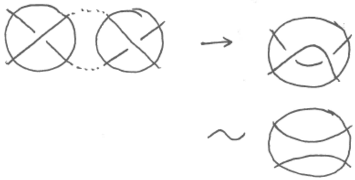
Like Twists Combine:

$$[1] + [1] = [2]$$

The number of twists is additive if they match.

Opposite Twists "Cancel":

$$[1] + [-1] \simeq [0]$$



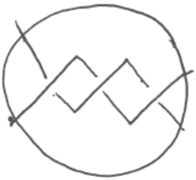
Opposite twists undo each other by an R2 move.

Integer Tangles

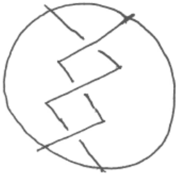
Definition

A **horizontal integer tangle** $T_a = [a]$ is a twist of two horizontal strands $|a|$ times in the direction of the sign of a.
A **vertical integer tangle** $T'_a = \left[\frac{1}{a}\right]$ is a twist of two vertical strands $|a|$ times in the direction of the sign of a.

Example: $T_3 = [3]$



Example: $T'_3 = \left[\frac{1}{3}\right]$



Rational Tangle General Definition: Inductive Construction

Definition

For any sequence of integers $a_1, \cdots, a_n$, choose a sequence of integer tangles $S_{a_1}, \cdots, S_{a_n}$, where either $S_{a_i} = T_{a_i}$ or $S_{a_i} = T'_{a_i}$. Define a tangle B_n by the following inductive construction.[5][6]

1. Let $B_1 = S_{a_1}$.
2. For $k < n$:
   i. If $S_{a_{k+1}} = T_{a_{k+1}}$, then either let $B_{k+1} = T_{a_{k+1}} + B_k$ or let $B_{k+1} = B_k + T_{a_{k+1}}$ (addition on left or right by $T_{a_{k+1}}$).
   ii. If $S_{a_{k+1}} = T'_{a_{k+1}}$, then either let $B_{k+1} = T'_{a_{k+1}} * B_k$ or let $B_{k+1} = B_k * T'_{a_{k+1}}$ (multiplication on top or bottom by $T'_{a_{k+1}}$).

Any tangle B_n which is constructed inductively by this algorithm is a **rational tangle** with n integer tangles.

[5] Jay R. Goldman and Louis H. Kauffman. Rational tangles. *Adv. in Appl. Math.*, 18(3):300-332, 1997

[6] J. H. Conway. An enumeration of knots and lints, and some of their algebraic properties. In *Computational Problems in Abstract Algebra (Proc. Conf., Oxford 1967)*, pages 329-358. Pergamon, Oxford, 1970

Rational Tangle General Definition: Explanation

A Rational Tangle Needs Two Things:

- an ordered list of integer tangle components $\{S_{a_1}, \cdots, S_{a_n}\}$
- information on how to combine these components

Rational Tangle General Definition: Example

Integer Tangles:

$$\begin{aligned} S_{a_1} &= T_3 \\ S_{a_2} &= T_2' \\ S_{a_3} &= T_2 \end{aligned}$$

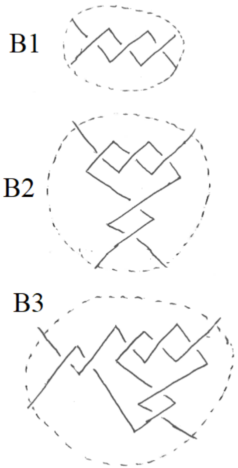
Construction Information:

$$\begin{aligned} B_1 &= S_{a_1} \\ &= T_3 \end{aligned}$$

$$\begin{aligned} B_2 &= B_1 * S_{a_2} \\ &= T_3 * T_2' \end{aligned}$$

$$\begin{aligned} B_3 &= S_{a_3} + B_2 \\ &= T_2 + (T_3 * T_2') \end{aligned}$$

Rational Tangle:



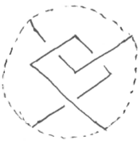
Rational Tangles in Canonical Form

Definition

A rational tangle is said to be in **canonical form** if

- it is an **alternating** tangle,
- it is constructed by alternating right and bottom twists.

Non-canonical:
$T_2 * T'_{-1}$

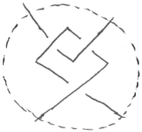


(non-alternating)

Non-canonical:
$T'_1 * T_2$

(twist on top)

Canonical:
$T_2 * T'_1$



Alternating Rational Tangles

Theorem

A rational tangle is alternating if and only if all twists have the same sign. A non-alternating rational tangle is reducible.

$$T_2 * T'_{-1} \simeq T_{-2}$$

Corollary

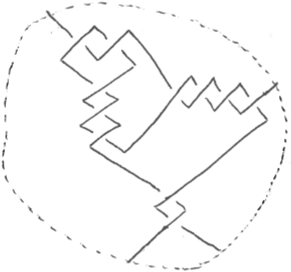
A canonical rational tangle has only positive or negative twists.

The Implicit Construction for the Canonical Form

Construction Simplifications:

- For the canonical form, we only need an ordered sequence of alternately horizontal and vertical integer tangles.
- Since twists must alternate, the construction is implied.

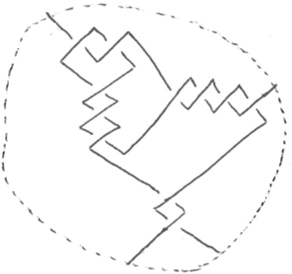
Example: $\{T_2, T'_3, T_4, T'_2\}$ $\qquad T = ((T_2 * T'_3) + T_4) * T'_2$



Twisting in Canonical Form

Example: $\{T_2, T_3', T_4, T_2'\}$ $\quad T = ((T_2 * T_3') + T_4) * T_2'$

- 2 horizontal twists
- 3 vertical twists
- 4 horizontal twists
- 2 vertical twists
- 0 horizontal twists



Compact Notation:

$$(2, 3, 4, 2, 0)$$

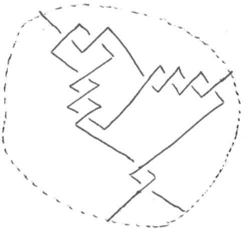
Since integer tangles must alternate horizontal and vertical, we only need the number and sign of each twist.

The Rational Twist Vector

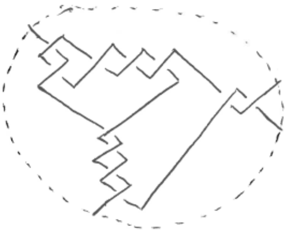
Vector Notation for Canonical Twisting:

- We represent the alternating twists with a vector.
- Each entry specifies a signed number of twists.
- **Convention:** The last entry denotes a horizontal twist.

Example: $(2, 3, 4, 2, 0)$



Example: $(2, 3, 4, 2)$



These are reflections across the NW-SE diagonal, but not isotopic.

The Rational Twist Vector: Formal Definition

Definition

Let T be a rational tangle in canonical form constructed from alternately horizontal and vertical integer tangles $\{S_{a_1}, \cdots, S_{a_n}\}$. Define the **canonical twist vector** $\mathbf{v}_T$ as follows.

- If $S_{a_n} = T_{a_n}$ (last twist is horizontal), then $\mathbf{v}_T = (a_1, \cdots, a_n)$.
- If $S_{a_n} = T'_{a_n}$ (last twist is vertical), then $\mathbf{v}_T = (a_1, \cdots, a_n, 0)$.

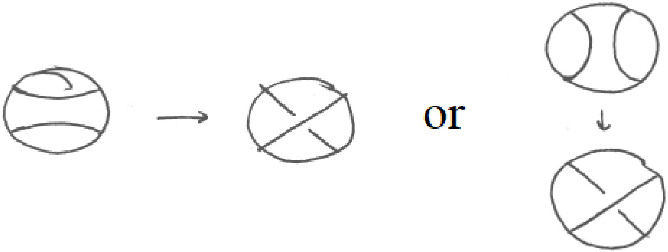
Properties:

- The twist vector explicitly defines the construction.
- The is a bijection between rational tangles and $\mathbb{Q} \cup \{\infty\}$, with a formula determined by a twist vector.

Twist Vector Uniqueness: First Twist

The very first twist in a rational tangle may be regarded as either:

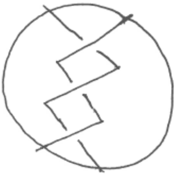
- a right twist of two horizontal strands; or
- a bottom twist of two vertical strands.



These define different twist vectors, but not different tangles.

Twist Vector Uniqueness: Example

Example: T_3'



$\mathbf{v}_{T_3'} = (1, 2, 0)$

- 1 horizontal twist right
- 2 vertical twists bottom
- 0 horizontal twists right

$\mathbf{v}_{T_3'} = (3, 0)$

- 3 vertical twists bottom
- 0 horizontal twists right

Convention:

- The first twist is always assumed to be horizontal.
- Therefore, twist vectors have odd length.

Operations on Rational Tangles[7][8]

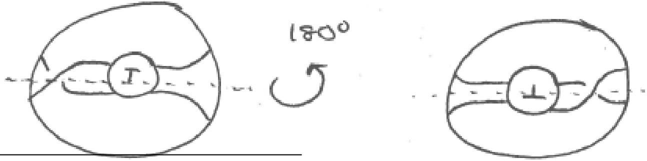
Theorem

A 180° rotation of a rational tangle T in the horizontal or vertical axis is ambient isotopic to T.

Corollary

If T is a rational tangle and $a \in \mathbb{Z}$ is an integer, then

$$T_a + T \simeq T + T_a \qquad \text{and} \qquad T'_a * T \simeq T * T'_a.$$



[7] Jay R. Goldman and Louis H. Kauffman. Rational tangles. *Adv. in Appl. Math.*, 18(3):300-332, 1997

[8] J. H. Conway. An enumeration of knots and lints, and some of their algebraic properties. In *Computational Problems in Abstract Algebra (Proc. Conf., Oxford 1967)*, pages 329-358. Pergamon, Oxford, 1970

Equivalence with Canonical Form

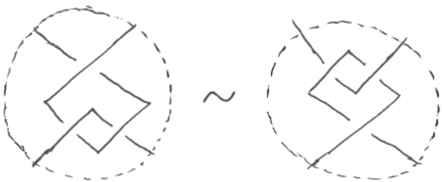
Theorem

Every rational tangle is equivalent to a tangle in canonical form.[9][10]

Intuitive Idea:

- Twists can be pushed into canonical position.
  - horizontal $\implies$ right
  - vertical $\implies$ bottom
- Rotate interior subtangle while fixing boundary.
- Each rotation moves one twist, repeat as needed.

Example: $T_1' * T_2 \simeq T_2 * T_1'$



[9] Jay R. Goldman and Louis H. Kauffman. Rational tangles. *Adv. in Appl. Math.*, 18(3):300-332, 1997

[10] J. H. Conway. An enumeration of knots and lints, and some of their algebraic properties. In *Computational Problems in Abstract Algebra (Proc. Conf., Oxford 1967)*, pages 329-358. Pergamon, Oxford, 1970

Equivalence with Canonical Form: Example

Example: $T'_{-2} * (T_{-2} + T'_{-3}) \simeq (T'_{-3} + T_{-2}) * T'_{-2}$

Original:

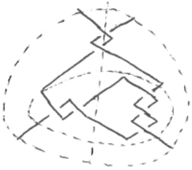
- -3 vertical twists bottom
- -2 horizontal twists left
- -2 vertical twists top

Canonical:

- -3 vertical twists bottom
- -2 horizontal twists **right**
- -2 vertical twists **bottom**

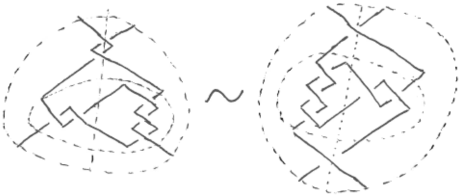
Equivalence with Canonical Form: Example: Detailed

Step 1

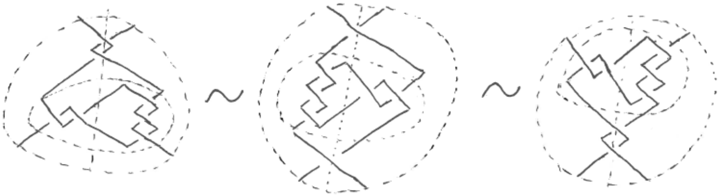


Equivalence with Canonical Form: Example: Detailed

Step 1



Equivalence with Canonical Form: Example: Detailed

Step 1 

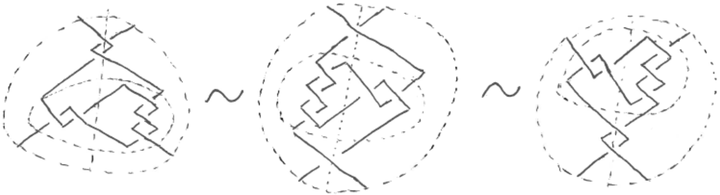
Equivalence with Canonical Form: Example: Detailed

Step 1

Step 2

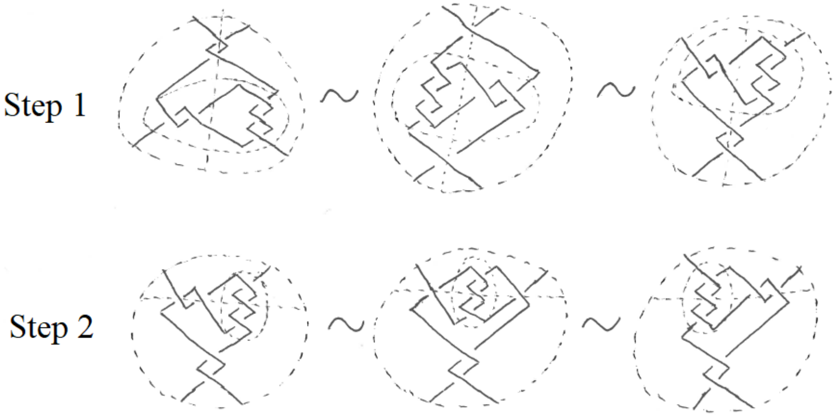
Equivalence with Canonical Form: Example: Detailed

Step 1



Step 2

Equivalence with Canonical Form: Example: Detailed



Bijection with the Extended Rational Numbers

Theorem

There is a bijective correspondence between rational tangles and the extended rational numbers.[11]

- Each twist vector defines a continued fraction in $\mathbb{Q} \cup \{\infty\}$.
- This fraction is a rational tangle invariant.

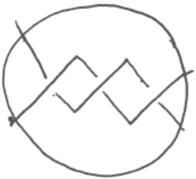
$$(a_1, a_2, \cdots, a_{n-1}, a_n) \qquad \leftrightarrow \qquad a_n + \cfrac{1}{a_{n-1} + \cdots + \frac{1}{a_2 + \frac{1}{a_1}}}$$

[11] J. H. Conway. An enumeration of knots and lints, and some of their algebraic properties. In *Computational Problems in Abstract Algebra (Proc. Conf., Oxford 1967)*, pages 329-358. Pergamon, Oxford, 1970

Special Case: Integer Tangles

Horizontal:

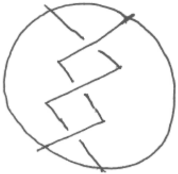
- Tangle: T_a
- Vector: $\mathbf{v}_{T_a} = (a)$
- Fraction: $\frac{p}{q} = \frac{a}{1}$



$T_3 = [3]$

Vertical:

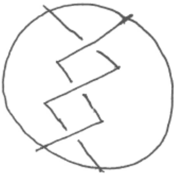
- Tangle: T'_a
- Vector: $\mathbf{v}_{T'_a} = (a, 0)$
- Fraction: $\frac{p}{q} = \frac{1}{a}$



$T'_3 = \left[\frac{1}{3}\right]$

Twist Vector Uniqueness: Continued Fraction

Example: $T_3' = \left[\frac{1}{3}\right]$



$\mathbf{v}_{T_3'} = (1, 2, 0)$

- 1 horizontal twist right
- 2 vertical twists bottom
- 0 horizontal twists right

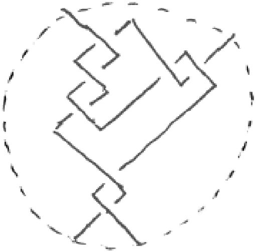
$\mathbf{v}_{T_3'} = (3, 0)$

- 3 vertical twists bottom
- 0 horizontal twists right

$$(1, 2, 0) \quad \mapsto \quad 0 + \frac{1}{2 + \frac{1}{1}} = \frac{1}{3}$$

$$(3, 0) \quad \mapsto \quad 0 + \frac{1}{3} = \frac{1}{3}$$

Bijection with the Extended Rational Numbers: Example



- -3 vertical twists
- -2 horizontal twists
- -2 vertical twists

Rational Tangle:

$T = (T'_{-3} + T_{-2}) * T'_{-2}$

Twist Vector:

$\mathbf{v}_T = (-1, -2, -2, -2, 0)$

Fraction:

$$\frac{p}{q} = 0 + \cfrac{1}{-2 + \cfrac{1}{-2 + \cfrac{1}{-2 + \frac{1}{-1}}}} = -\frac{7}{17}$$

Rational Tangle Construction Algorithm: Overview

Goal: Exhaustively construct all rational tangles.

- Find rational tangles up to specified crossing number.
- Need only consider canonical form.
- Obtain explicit Gauss code representation.

Method: Generate tangles computationally with a program in C.

- Exploit correspondence between tangles and twist vectors.
- Find exhaustive list of twist vectors.
- Reconstruct each tangle from a twist vector.
- Calculate corresponding extended rational number.

Twist Vectors and Integer Compositions

Let T be a rational tangle with k crossings and twist vector $\mathbf{v}_T = (a_1, \cdots, a_n)$. Note that:

$$k = |a_1| + \cdots + |a_n|$$

Accounting for twist sign, the number of possible expressions for $\mathbf{v}_T$ is twice the number of integer compositions of k.

Definition

Given a positive integer k, a **composition** of k is an ordered sequence of positive integers which sum to k.

In general, there are 2^{k-1} distinct compositions of k.

Generating Twist Vectors via Integer Compositions

k	composition	$\mathbf{v}_T$
0	0	(0)
1	1	(1)
2	2	(2)
	1 + 1	(1, 1, 0)
3	3	(3)
	2 + 1	(2, 1, 0)
	1 + 2	(1, 2, 0)
	1 + 1 + 1	(1, 1, 1)

k	composition	$\mathbf{v}_T$
4	4	(4)
	3 + 1	(3, 1, 0)
	2 + 2	(2, 2, 0)
	2 + 1 + 1	(2, 1, 1)
	1 + 3	(1, 3, 0)
	1 + 2 + 1	(1, 2, 1)
	1 + 1 + 2	(1, 1, 2)
	1 + 1 + 1 + 1	(1, 1, 1, 1, 0)

$\vdots$

Computational Construction Algorithm

Step 1: Twist Vectors

- Choose a maximum crossing number N.
- Generate integer compositions of k, for $1 \leq k \leq N$.
- Convert compositions to twist vectors, including signs.

Step 2: Rational Tangles

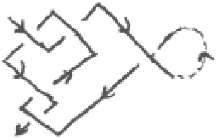
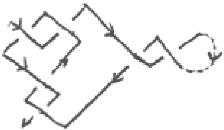
- Iterate through each twist vector.
- Convert twist vector to Gauss code for a rational tangle.
- Build up the code one crossing at a time.

Construction Algorithm: Example: $(-2,-2,-2)$

$$\mathbf{v}_T = (-2,-2,-2) \text{ gives } T = (T_{-2} * T'_{-2}) + T_{-2} = \left[-\tfrac{12}{5}\right]$$

twist	diagram	Gauss code
1		$\|)\underline{a1-}(\|)\underline{b1-}(\|$
2		$\|)a1-\underline{b2-}(\|)\underline{a2-}b1-(\|$
3		$\|)a1+b2+(\|)\underline{a3+}b1+a2+\underline{b3+}(\|$

Construction Algorithm: Example: $(-2, -2, -2)$

twist	diagram	Gauss code
4		$|)a1 - b2 - (|)$ $\underline{a3+}b4 + a2 - b1 - a4 + \underline{b3+}(|$
5		$|)a1 - b2 - \underline{a3-}(|)\underline{b3-}$ $a4 + b5 + a2 - b1 - a5 + b4 + (|$
6		$|)a1 - b2 - a3 - \underline{b4-}(|)\underline{a4-}b3-$ $a5 + b6 + a2 - b1 - a6 + b5 + (|$

Constructing a Rational Tangle from a Twist Vector

Structural Changes:

- Each step adds a twist to the diagram of the previous step.
  - **Horizontal:** new crossing at right endpoints
  - **Vertical:** new crossing at bottom endpoints
- The Gauss code only changes slightly between steps.
  - **Horizontal:** last crossing string 1, first crossing string 2
  - **Vertical:** first crossing string 2, last crossing string 2

Consequences:

- Crossing number increases
- Possible parity change
- Orientation reversal on string 2 (if vertical)
  - sign reversal on shared crossings
- Crossing order change

Detecting Rational Tangles in General

Reverse Question:
Given an arbitrary 2-string tangle, how do we know if it's rational?

Solution:

- A tangle is rational if and only if it is built by twisting.
- Endpoint twisting is easy to detect from the Gauss code.
- Reversing the construction, we can unwind twists.
- If all twists can be unwound, the tangle is rational.
- The twist vector is determined through unwinding.

Connection with Main Project:
An unwinding subroutine is now included with the main project!

Table of Contents

Hierarchy of 2-String Tangle Types

$$\begin{array}{c}
\{\text{arbitrary 2-string tangles}\} \\
\cup\!\mid \\
\{\text{algebraic tangles}\} \\
\cup\!\mid \\
\{\text{generalized Montesinos tangles}\} \\
\cup\!\mid \\
\{\text{rational tangles}\} \\
\cup\!\mid \\
\{\text{integer tangles}\}
\end{array}$$

Generalized Montesinos Tangles

Definition

Informally, a **generalized Montesinos tangle** can be thought of as a core subtangle with endpoint twisting.

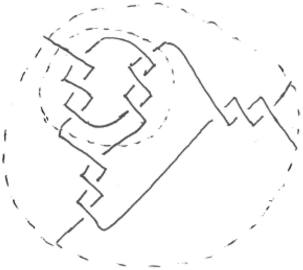
- The core consists of a finite sum of rational tangles.
- The endpoint twisting can be described with a twist vector.

- A **family** of tangles with representative T refers to the collection of generalized Montesinos tangles with core T.
- The rational tangles comprise the family of the 0-tangle.

Generalized Montesinos Tangles: Example

Example:
$T'_{-3} + T'_{-3}$

Example:
$(T'_{-3} + T'_{-3}) \circ (-3, -3)$



Example of two non-rational Montesinos tangles in the same family.

Algebraic Tangles

Definition

A tangle is **algebraic** if it can be expressed with a finite combination of sums and products of rational tangles.

Example:

$$(T_3' + T_2') * (T_2' + T_3')$$

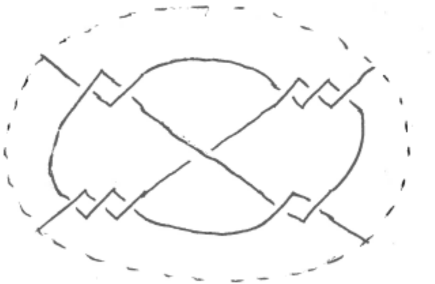
Example of a non-Montesinos algebraic tangle.

Non-Algebraic Tangles

Definition

A tangle is **non-algebraic** if it cannot be expressed in terms of a finite combination of sums and products of rational tangles.

Example:

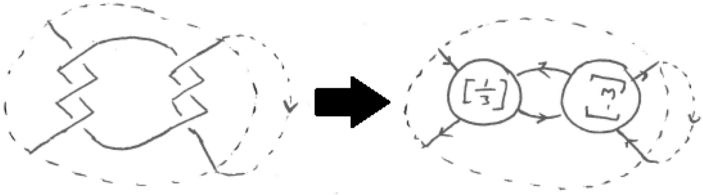


Adapting Conway's Idea

Observations:

- Rational tangles are well-behaved and easy to describe.
- Rational tangles "live inside" non-rational tangles.

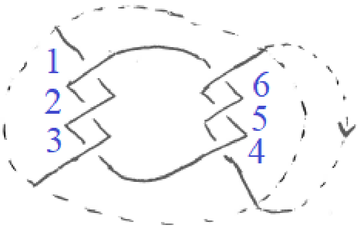
Idea: Can we use rational subtangles rather than crossings?



Recall: Planar Diagram Code

Example: $T = T_3' + T_3'$

$+$	$2b$	$3a$	$6d$	$2c$
$+$	$3b$	$1a$	$1d$	$3c$
$+$	$1b$	$2a$	$2d$	$4c$
$+$	$5b$	$6a$	$3d$	$5c$
$+$	$6b$	$4a$	$4d$	$6c$
$+$	$4b$	$5a$	$5d$	$1c$

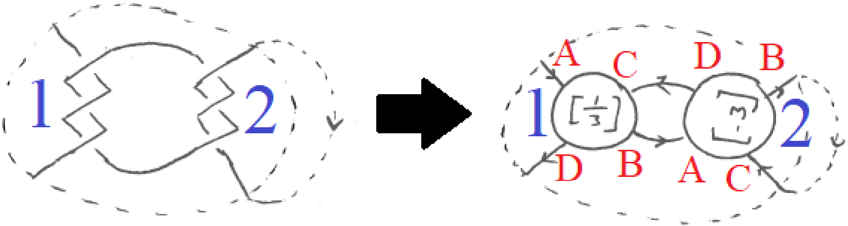


Tangles are described by their crossing connections.

Generalizing the Planar Diagram Code

Example: $T = T_3' + T_3'$

$$\begin{array}{llllllll} (1/3) & 1D & 2A & 2D & 1A & (\text{Parity}: 1) & \text{Twist Vector:} & (1\ \ 2\ \ 0) \\ (3/-1) & 1B & 2C & 2B & 1C & (\text{Parity}: 1') & \text{Twist Vector:} & (-3) \end{array}$$



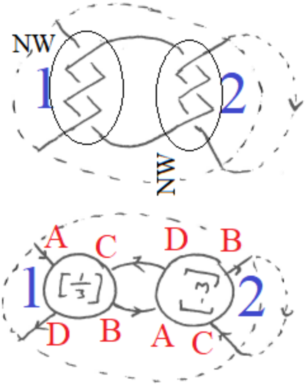
Tangles are described by their subtangle connections.

Generalized PD Code: Subtangle Labeling Convention

New Rules for Subtangles:

- Each subtangle has two strings: $S1$ and $S2$.
- Each corner matches a string entrance or exit.
- Labeling rule:
  - $A = S1$ entrance
  - $B = S1$ exit
  - $C = S2$ entrance
  - $D = S2$ exit
- Identify A as NW.
- Denote subtangle by corresponding rational p/q.

Example: $T = T_3' + T_3'$



Generalized PD Code: Additional Information

Displayed Subtangle Information:

- Each subtangle corresponds to one row.
- Each row displays the following subtangle information:
    - Corresponding rational number p/q
    - Subtangle connection information
    - Subtangle parity
        - internal string direction
    - Rational twist vector
- A "subtangle Gauss code" variant is tracked separately.

Generalized PD Code: Additional Information

Displayed Subtangle Information:

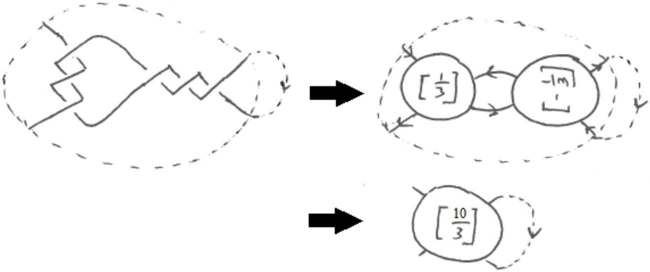
- Each subtangle corresponds to one row.
- Each row displays the following subtangle information:
  - Corresponding rational number p/q
  - Subtangle connection information
  - Subtangle parity
    - internal string direction
  - Rational twist vector
- A “subtangle Gauss code” variant is tracked separately.

Benefits of the New Notation:

- Rational tangles collapse to a single row.
  - This notation immediately identifies rational tangles.
- Two shared connections indicates a sum or product.
  - This suggests a criterion for classifying algebraic tangles.
- Conway would be proud.

Generalized PD Code: Rational Example

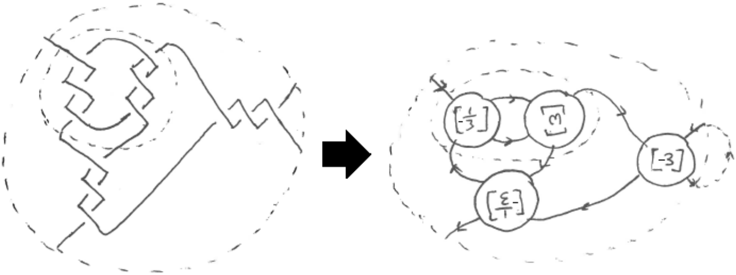
Example: $T = T_3' + T_3$



$(10/3) \quad 1D \quad 1C \quad 1B \quad 1A \quad (\text{Parity}: 0) \quad$ Twist Vector: $\quad (1 \;\; 2 \;\; 3)$

Generalized PD Code: Generalized Montesinos Example

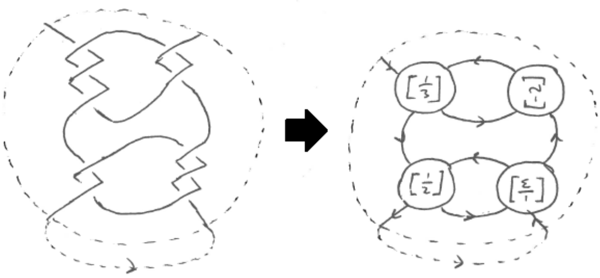
Example: $T = (T'_{-3} + T'_{-3}) \circ (-3, -3)$



$(1/-3)$	$4D$	$2A$	$4B$	$2C$	(Parity : $1'$)	Twist Vector:	$(-1 \quad -2 \;\; 0)$
$(3/1)$	$1B$	$3A$	$1D$	$4C$	(Parity : 1)	Twist Vector:	(3)
$(3/-1)$	$2B$	$3C$	$3B$	$4A$	(Parity : 1)	Twist Vector:	(-3)
$(1/-3)$	$3D$	$1C$	$2D$	$1A$	(Parity : $1'$)	Twist Vector:	$(-1 \quad -2 \;\; 0)$

Generalized PD Code: Algebraic Example

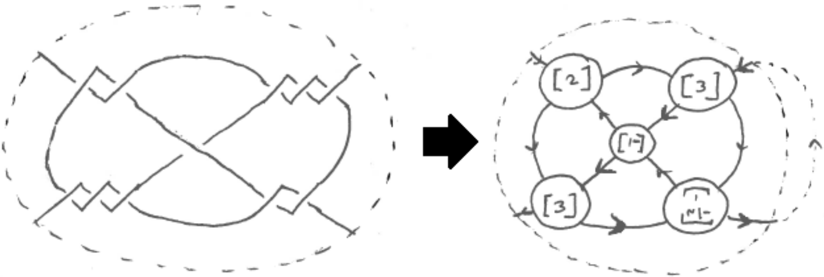
Example: $T = (T_3' + T_2') * (T_2' + T_3')$



$(1/3)$	$2D$	$2A$	$2B$	$3A$	(Parity : 1)	Twist Vector:	$(1\ \ 2\ \ 0)$
$(2/-1)$	$1B$	$1C$	$4D$	$1A$	(Parity : $0'$)	Twist Vector:	(-2)
$(1/-2)$	$1D$	$4A$	$4B$	$4C$	(Parity : ∞')	Twist Vector:	$(-1\ \ -1\ \ 0)$
$(1/3)$	$3B$	$3C$	$3D$	$2C$	(Parity : 1)	Twist Vector:	$(1\ \ 2\ \ 0)$

Generalized PD Code: Non-Algebraic Example

Example:



$(2/1)$	$5D$	$2A$	$4B$	$5A$	(Parity : 0)	Twist Vector:	(2)
$(3/1)$	$1B$	$3A$	$3D$	$4C$	(Parity : 1)	Twist Vector:	(3)
$(1/-2)$	$2B$	$4A$	$5B$	$2C$	(Parity : ∞)	Twist Vector:	$(-1 \quad -1 \quad 0)$
$(1/-1)$	$3B$	$1C$	$2D$	$5C$	(Parity : $1'$)	Twist Vector:	(-1)
$(3/1)$	$1D$	$3C$	$4D$	$1A$	(Parity : 1)	Twist Vector:	(3)

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