Constellations and an Algebraic Planar Diagram Code

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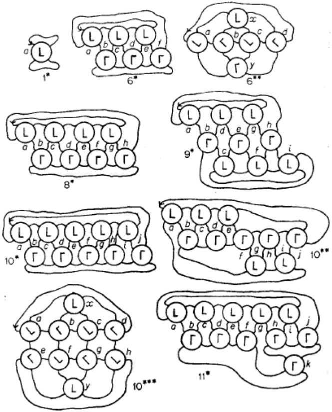
Connection to Knot Theory: Conway's Tabulation[1]

- Conway introduced tangles as a way to tabulate knots.
- Tangles are regarded as portions of knot diagrams.
  - planar polyhedra describe diagrams
  - vertices represent tangles
- All knots up to 11 crossings can be described this way.
  - basic polyhedra (right)
  - algebraic tangle vertices
- Improved on by Caudron.



[1] J. H. Conway. An enumeration of knots and lints, and some of their algebraic properties. In *Computational Problems in Abstract Algebra (Proc. Conf., Oxford 1967)*, pages 329-358. Pergamon, Oxford, 1970

Tangle Diagrams and Parity

Definition

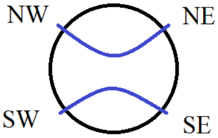
A **2-string tangle** consists of ball with two entwined strings embedded inside (their endpoints lie on the surface).

A **tangle diagram** is a 2-dimensional projection of a tangle.

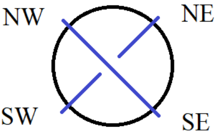
- Crossings distinguish over-strand and under-strand.
- Endpoints are identified by NW, NE, SE, SW (2-string).

A 2-string tangle has three possible **parities**.

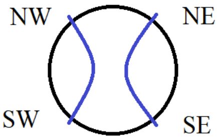
Parity 0:



Parity 1:



Parity ∞:

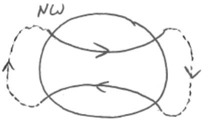


Tangle Properties Similar to Knots

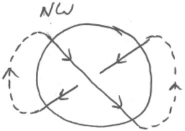
Definition

An **orientation** on a tangle is a choice of direction for each string.

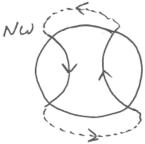
Parity 0:



Parity 1:



Parity ∞:



By convention, each parity type has a standard orientation.

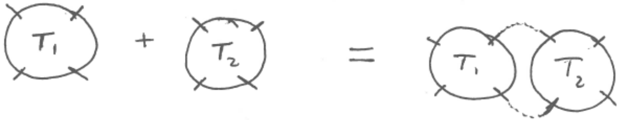
Definition

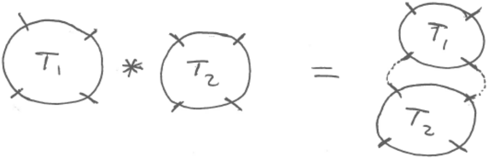
Two tangles are said to be **equivalent** if there exists an ambient isotopy between them which leaves the boundary of the ball fixed. Equivalent diagrams are related by **Reidemeister moves**.

Sum and Product of 2-String Tangles

Definition

A **sum** of tangles $T_1 + T_2$ joins the tangles horizontally as shown.
A **product** of tangles $T_1 * T_2$ joins the tangles vertically as shown.





Notation Examples

Dowker Code:

$$-4 \mid -6 \quad -2$$

Gauss Code:

$$|)a1+b2+(|)a3+b1+a2+b3+(|$$

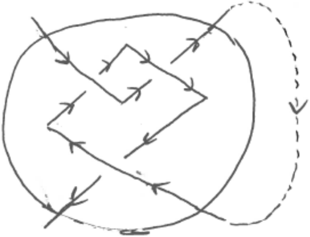
Planar Diagram Code:

$$\begin{array}{ccccc} + & b2 & a3 & d3 & c2 \\ + & b3 & a1 & d1 & c3 \\ + & b1 & a2 & d2 & c1 \end{array}$$

Coloring Matrix:

$$\begin{bmatrix} 1 & -2 & 0 & 1 & 0 \\ -2 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & -2 & 1 \end{bmatrix}$$

Example:

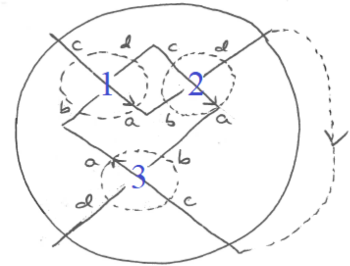


Planar Diagram Code

Algorithm:

Example:

$$
\begin{array}{ccccc}
+ & b2 & a3 & d3 & c2 \\
+ & b3 & a1 & d1 & c3 \\
+ & b1 & a2 & d2 & c1
\end{array}
$$



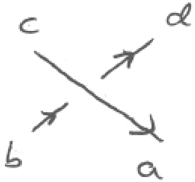
Planar Diagram Code: Labeling Convention

Crossing Labeling Convention:

- Each crossing has four corners.
- Label outward pointing overstrand a.
- Label remaining corners clockwise by b, c, d.

Corner Connections:

Example:



Planar Diagram Code: Labeling Convention

Crossing Labeling Convention:

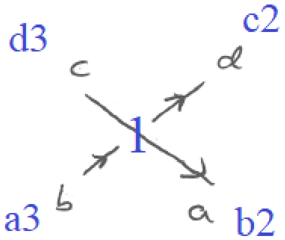
- Each crossing has four corners.
- Label outward pointing overstrand *a*.
- Label remaining corners clockwise by *b*, *c*, *d*.

Corner Connections:

- Index each crossing.
- Record crossing sign.
- For each corner, record connected crossing.

$$\underbrace{+}_{\text{sign}}\ \underbrace{b2}_{a}\ \underbrace{a3}_{b}\ \underbrace{d3}_{c}\ \underbrace{c2}_{d}$$

Example:



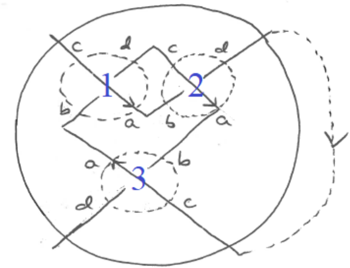
Planar Diagram Code: Algorithm

Algorithm:

1. Start at NW corner.
2. Index crossings with subsequent integers.
3. Label corners of each crossing a, b, c, d.
4. Record corner connections for each crossing.
5. List rows with corner connection information.

Example:

1	$+$	$b2$	$a3$	$d3$	$c2$
2	$+$	$b3$	$a1$	$d1$	$c3$
3	$+$	$b1$	$a2$	$d2$	$c1$



Hierarchy of 2-String Tangle Types

$$\{\text{arbitrary 2-string tangles}\}$$
$$\cup\!\mid$$
$$\{\text{algebraic tangles}\}$$
$$\cup\!\mid$$
$$\{\text{generalized Montesinos tangles}\}$$
$$\cup\!\mid$$
$$\{\text{Montesinos tangles}\}$$
$$\cup\!\mid$$
$$\{\text{rational tangles}\}$$
$$\cup\!\mid$$
$$\{\text{integer tangles}\}$$

2-String Tangle Types by Increasing Complexity

Integer

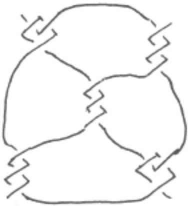
Rational

Montesinos

Generalized Montesinos

Algebraic

Non-Algebraic

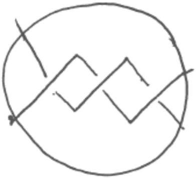


Integer Tangles

Definition

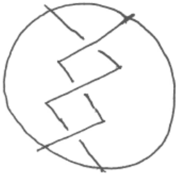
An **integer tangle** is constructed from any sequence of horizontal or vertical twists.

Example: $3/1$



horizontal

Example: $1/3$



vertical

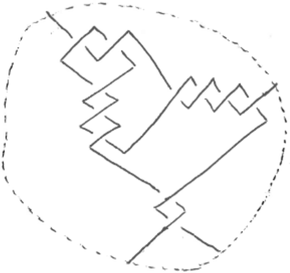
Rational Tangles

Definition

A **rational tangle** is constructed from an alternating sequence of horizontal and vertical twists, denoted by a **twist vector**.

Example: rational tangle with canonical* diagram

- 2 horizontal twists
- 3 vertical twists
- 4 horizontal twists
- 2 vertical twists
- 0 horizontal twists



Twist Vector:

$$(2, 3, 4, 2, 0)$$

Rational Tangles: Bijection with Extended Rationals

Theorem

Rational tangles (up to equivalence with canonical form) are in bijection with the extended rational numbers.[2]

- Every rational tangle can be placed in canonical form.
- Every canonical form can be described with a twist vector.
- Each twist vector defines a continued fraction in $\mathbb{Q} \cup \{\infty\}$.

$$(a_1, a_2, \cdots, a_{n-1}, a_n) \qquad \leftrightarrow \qquad a_n + \cfrac{1}{a_{n-1} + \cdots + \frac{1}{a_2 + \frac{1}{a_1}}}$$

- This fraction is a rational tangle invariant.

[2] J. H. Conway. An enumeration of knots and lints, and some of their algebraic properties. In *Computational Problems in Abstract Algebra (Proc. Conf., Oxford 1967)*, pages 329-358. Pergamon, Oxford, 1970

Rational Tangles: Example

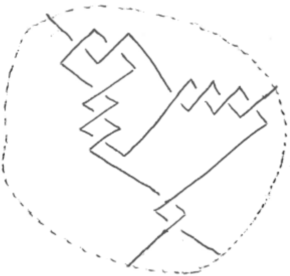
Twist Vector:

$$(2, 3, 4, 2, 0)$$

Fraction:

$$0 + \cfrac{1}{2 + \cfrac{1}{4 + \cfrac{1}{3 + \frac{1}{2}}}} = \frac{30}{67}$$

Diagram:

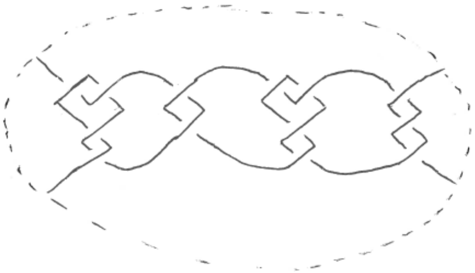


Montesinos Tangles

Definition

A tangle is **Montesinos** if it can be expressed as a horizontal or vertical joining of rational tangles.

Example: $(2/5) + (1/2) + (2/5) + (1/3)$

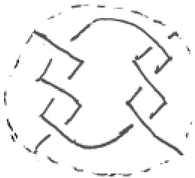


Generalized Montesinos Tangles

Definition

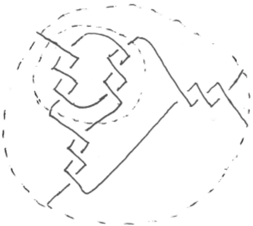
A **generalized Montesinos tangle** consists of a core Montesinos subtangle with external twisting.

Example: Montesinos Core



$$\left(\frac{1}{-3}\right) + \left(\frac{1}{-3}\right)$$

Example: Endpoint Twisting



$$\left(\frac{1}{-3}\right) + \left(\frac{1}{-3}\right) \circ (-3, -3, 0)$$

Database of 2-String Tangles

Prototype of database of 2-string tangles:

- http://www.nick-connolly.com/tangles
- rational tangles up to 9 crossings
- generalized Montesinos tangles up to 11 crossings

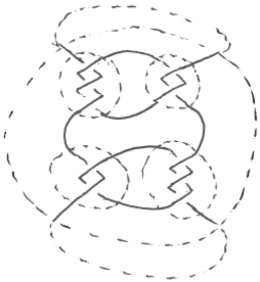
Algebraic Tangles

Definition

A tangle is **algebraic** if it can be expressed with a finite combination of sums and products of rational tangles.

Example:

$$((1/3)+(1/2))*((1/2)+(1/3))$$



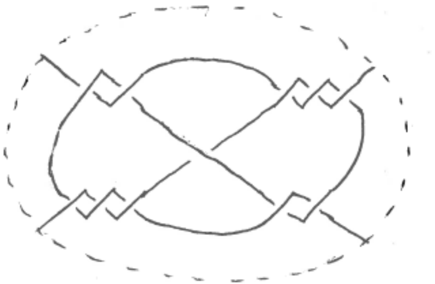
non-Montesinos algebraic tangle

Non-Algebraic Tangles

Definition

A tangle is **non-algebraic** if it cannot be expressed in terms of a finite combination of sums and products of rational tangles.

Example:



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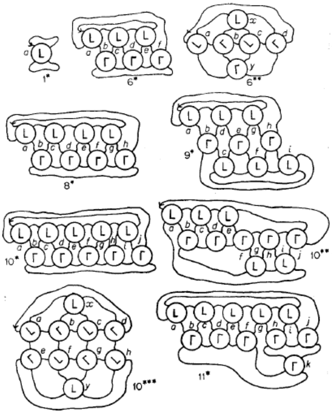
Recall: Conway's Tabulation[3]



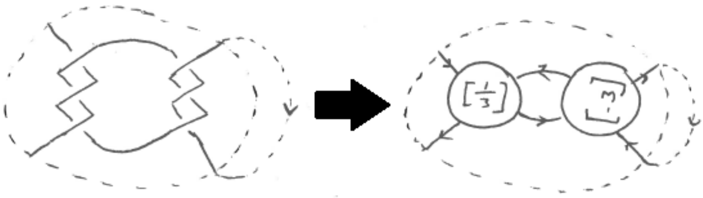
[3] J. H. Conway. An enumeration of knots and lints, and some of their algebraic properties. In *Computational Problems in Abstract Algebra (Proc. Conf., Oxford 1967)*, pages 329-358. Pergamon, Oxford, 1970

Adapting Conway's Idea

Observations:

- Rational tangles are well-behaved and easy to describe.
- Rational tangles "live inside" non-rational tangles.

Idea: Can we use rational subtangles rather than crossings?

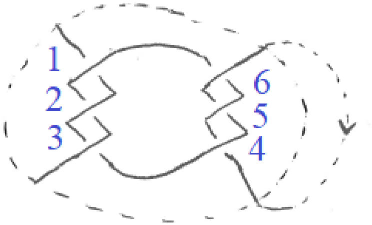


Recall: Planar Diagram Code

The PD code describes a graph using crossing connections:

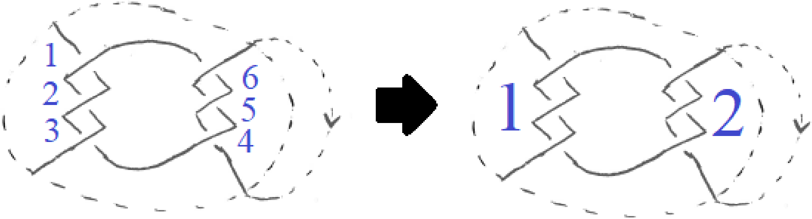
- vertices are crossings
- edges are connecting strands
- crossing sign is tracked

Example: $(1/3) + (1/3)$



$+$	$2b$	$3a$	$6d$	$2c$
$+$	$3b$	$1a$	$1d$	$3c$
$+$	$1b$	$2a$	$2d$	$4c$
$+$	$5b$	$6a$	$3d$	$5c$
$+$	$6b$	$4a$	$4d$	$6c$
$+$	$4b$	$5a$	$5d$	$1c$

Generalize to a Subtangle Diagram Code?



Original PD Code

$+$	$2b$	$3a$	$6d$	$2c$
$+$	$3b$	$1a$	$1d$	$3c$
$+$	$1b$	$2a$	$2d$	$4c$
$+$	$5b$	$6a$	$3d$	$5c$
$+$	$6b$	$4a$	$4d$	$6c$
$+$	$4b$	$5a$	$5d$	$1c$

Subtangle Diagram Code?

?	$1d$	$2a$	$3d$	$1a$	?
?	$1b$	$2c$	$2b$	$1c$	?

How to Make Rigorous?

Goal: Define a PD notation using substangles

Considerations:

- What kind of subtangles?
- How to label corners?
- How to handle endpoints?
- How to deal with equivalent subtangles?
- How to leverage graph theory?
- How to compute notation?
- How to implement with programs?

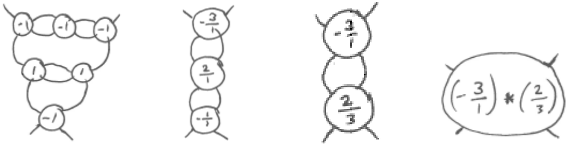
Other Goal: Distinguish non-algebraic tangles with notation

Subtangle Choices

Subtangle crossings integer rational algebraic

Diagram

Graph



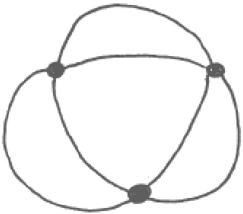
The Graph of a Knot Diagram

Any knot diagram can be turned into a graph.

- The crossings become vertices.
- The strands between crossings become edges.

Example: Trefoil Diagram

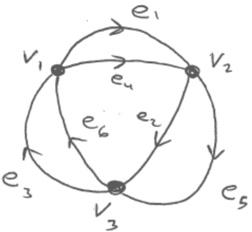
Example: Trefoil Graph



Knot Orientation and Eulerian Circuits

Observe that a choice of orientation on a knot diagram induces an Eulerian circuit on the corresponding knot graph.

Example: Oriented Trefoil



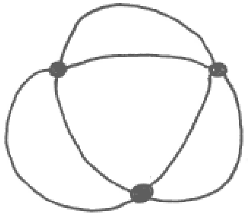
$$s = [v_1, e_1, v_2, e_2, v_3, e_3, v_1, e_4, v_2, e_5, v_3, e_6, v_1]$$

Properties of a Knot Diagram Graph

Properties:

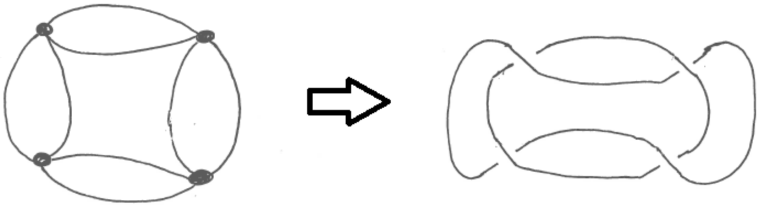
- The graph is **connected**.
- The graph is **planar**.
- The graph is **4-regular** (each vertex has degree 4).
- The graph contains an **Eulerian circuit**.

Graph:



Reverse Question: Building Knot Diagrams from Graphs

Any connected, planar, 4-regular graph can be realized as a link.



Graph

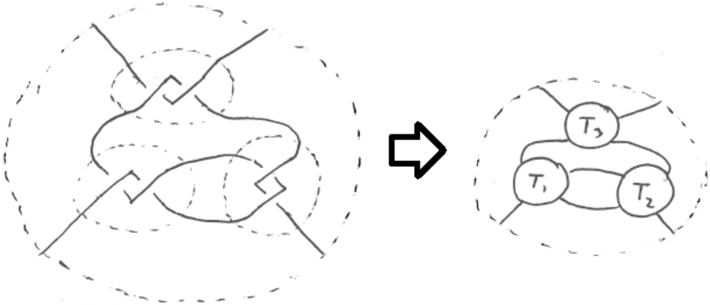
Diagram

A graph with n vertices can be realized by 2^n diagrams.

Describing Tangles with Graphs

Replacing subtangles with vertices gives another graph theoretic description of a diagram.

Example: $T_3 * (T_1 + T_2)$

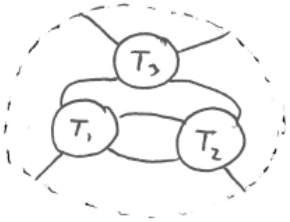


Subtangle Graphs

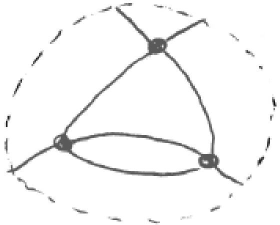
We can also model these subtangle combinations with a graph.

- The subtangles becomes vertices.
- The strands between subtangles become edges.

Example: Tangle Diagram



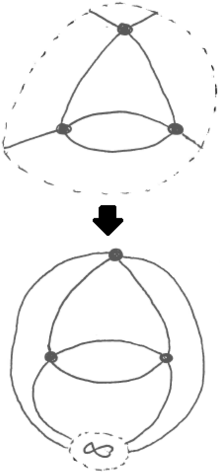
Example: Subtangle "Graph"



Properties of a Subtangle Diagram Graph

Properties:

- The graph is **connected**.
- The graph is **planar**.
- The graph is **4-regular** (each vertex has degree 4).
- The graph contains an **Eulerian circuit**.
- The endpoints form an extra **vertex at** ∞.

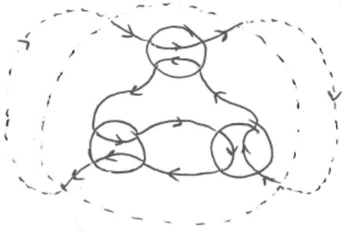


Eulerian Circuits and Subtangle Parities

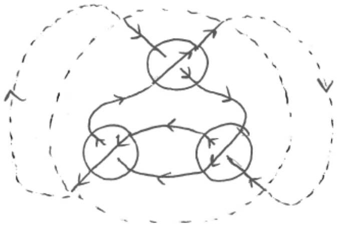
Theorem

A choice of Eulerian circuit in a subtangle diagram graph uniquely induces a parity on each subtangle component.

Example: Parities 0, 0, ∞



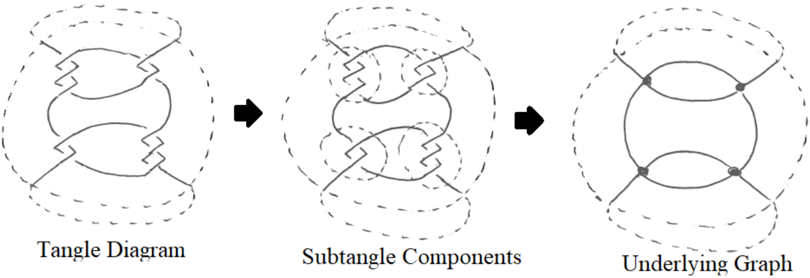
Example: Parities 1, 1, 1



Finding possible parities is equivalent to finding Eulerian circuits.

Reversing the Question

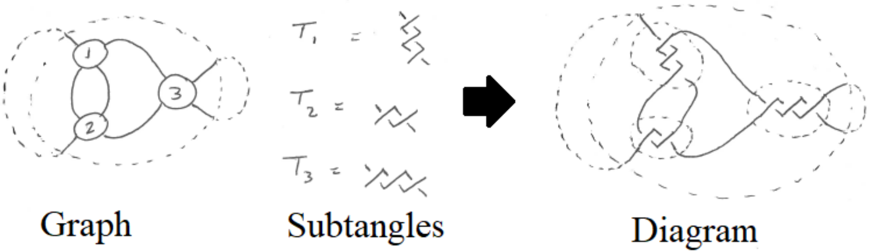
Every tangle diagram with subtangle components identified defines a graph with these properties.



Does every graph with these properties define a diagram?

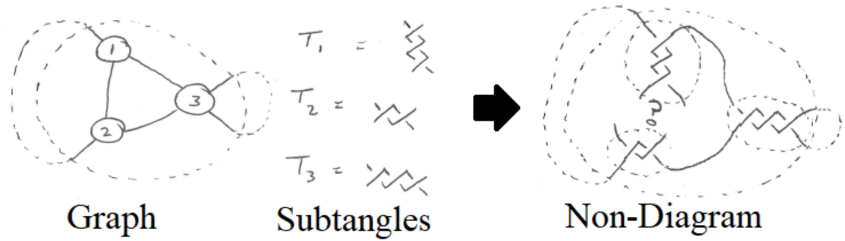
Building a Diagram from a Graph

Given a graph with these properties, we can build up a configuration of subtangles. Filling these in gives a tangle diagram.



Necessary Assumptions

Notice that we cannot connect subtangles without 4-regularity.



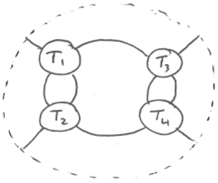
Can we add more restrictions to distinguish types of diagrams?

Algebraic Diagrams

Definition

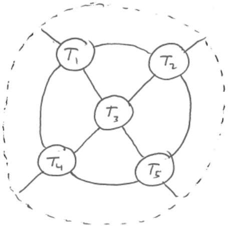
A subtangle diagram is **algebraic** if it can be built from sums and products of subtangles.

Example: Algebraic



$(T_1 * T_2) + (T_3 * T_4)$

Example: Non-Algebraic

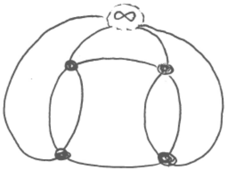


Tangle sums and products always join two endpoints at a time.

Graphs of Algebraic Diagrams

In the corresponding subtangle graph, there are two edges between added/multiplied subtangles (it's a **multi-graph**).

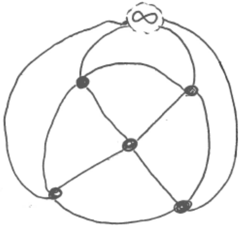
Example: Algebraic



$$(T_1 * T_2) + (T_3 * T_4)$$

Graph has multi-edges

Example: Non-Algebraic



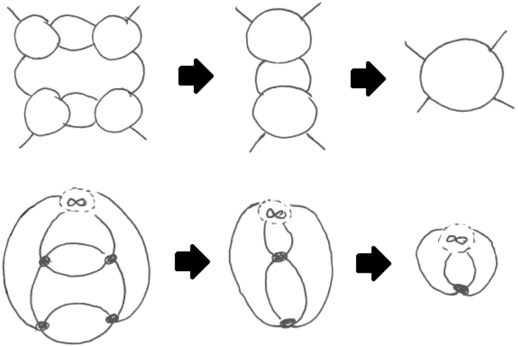
Graph is simple

The Maximal Algebraic Subtangle Graph

Definition

The **maximal algebraic subtangle graph** of a tangle diagram is obtained by collapsing together any vertices with two connections.

Example:



Distinguishing Algebraic and Non-Algebraic Diagrams

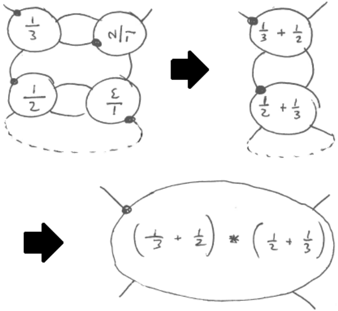
Theorem

A tangle is algebraic if and only if its **maximal algebraic subtangle graph** *consists of a single subgtangle vertex.*

Example: Algebraic Tangle

$((1/3)+(1/2))*((1/2)+(1/3))$

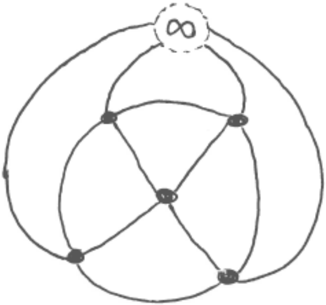
MASG:

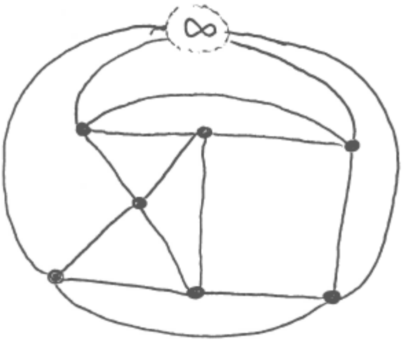


When is a Diagram Non-Algebraic?

Theorem

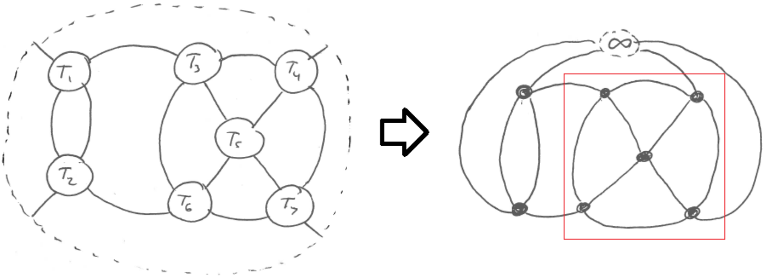
A **simple** *subtangle graph must be non-algebraic.*





Non-Algebraic Condition: Sufficient but not Necessary

Simple subtangle graphs must be non-algebraic; the converse does **not** hold. A non-algebraic subtangle graph isn't always simple.

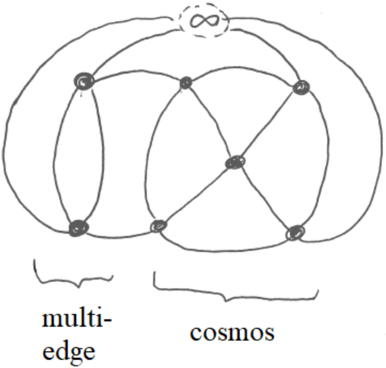


However, this graph contains a particular simple **subgraph**.

Constellations and Non-Algebraic Subtangle Graphs

Theorem

A subtangle graph is non-algebraic if and only if it contains a subgraph which is isomorphic to the cosmos of some constellation.



Generalizing the Planar Diagram Code

Standard Planar Diagram Code:

- Rows represent crossings.
- Precede row with crossing sign.
- Connection labels follow outward-pointing overstrand.

$$+ \quad 2b \quad 3a \quad 6d \quad 2c$$

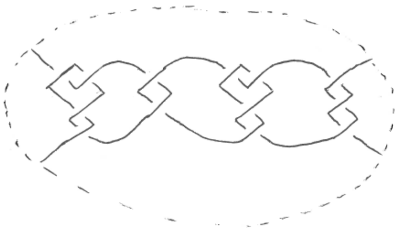
Integer/Rational/Algebraic Planar Diagram Code:

- Rows represent subtangles.
- Precede row with subtangle fraction/construction.
- Connection labels follow compass directions.
- Succeed row with subtangle parity.

$$(2/3) + (1/3) \quad 2\ \mathrm{NE} \quad 3\ \mathrm{NW} \quad 6\ \mathrm{SW} \quad 2\ \mathrm{SE} \quad (\mathrm{P}: 0')$$

Standard Planar Diagram Code: Example

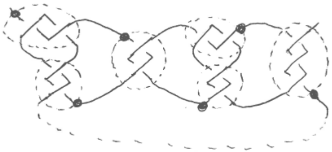
$$(2/5) + (1/2) + (2/5) + (1/3)$$

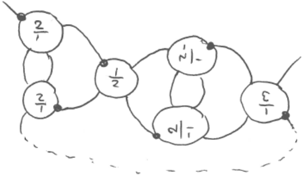


$+$	$6d$	$9a$	$2d$	$2c$
$+$	$3b$	$6a$	$1d$	$1c$
$+$	$4b$	$2a$	$11d$	$4c$
$+$	$5d$	$3a$	$3d$	$12c$
$-$	$7b$	$6c$	$6b$	$4a$
$-$	$2b$	$5c$	$5b$	$1a$
$+$	$8b$	$5a$	$12b$	$8c$
$+$	$9b$	$7a$	$7d$	$9c$
$+$	$1b$	$8a$	$8d$	$10c$
$+$	$11b$	$11a$	$9d$	$13c$
$+$	$10b$	$10a$	$13b$	$3c$
$-$	$13d$	$7c$	$4d$	$13a$
$-$	$12d$	$11c$	$10d$	$12a$

Integer Planar Diagram Code: Example

Example: $(2/5) + (1/2) + (2/5) + (1/3)$

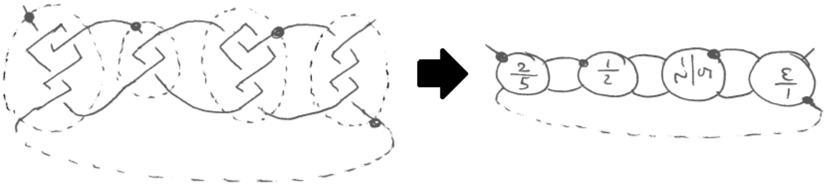




$(2/1)$	0 NW	2 NW	3 SW	3 SE	(P:0)
$(1/2)$	1 NE	5 SW	6 NW	3 NW	$(P{:}\infty')$
$(1/2)$	2 SW	0 SW	1 SW	1 SE	$(P{:}\infty)$
$(1/3)$	0 SE	6 SW	5 NW	0 NE	(P:1)
$(1/-2)$	4 SE	6 SE	6 NE	2 NE	$(P{:}\infty)$
$(2/-1)$	2 SE	5 SE	5 NE	4 NE	(P:0)

Rational Planar Diagram Code: Example

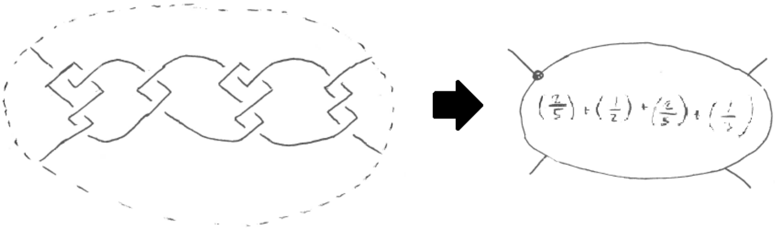
Example: $(2/5) + (1/2) + (2/5) + (1/3)$



$(2/5)$	0 NW	2 NW	2 SW	0 SW	(P:0)
$(1/2)$	1 NE	4 SW	4 SE	1 SE	(P:∞')
$(1/3)$	0 SE	4 NE	4 NW	0 NE	(P:1)
$(5/-2)$	3 SE	3 NE	2 SE	2 NE	(P:∞)

Algebraic Planar Diagram Code: Example

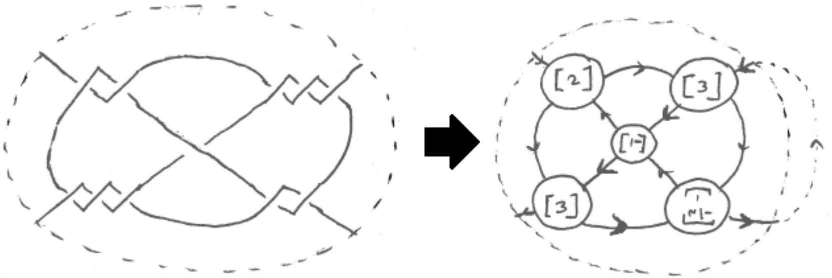
Example: $(2/5)+(1/2)+(2/5)+(1/3)$



$(2/5)+(1/2)+(2/5)+(1/3)$ 0 NW 0 NE 0 SE 0 SW $(P{:}\infty)$

Algebraic Planar Diagram Code: Non-Algebraic Example

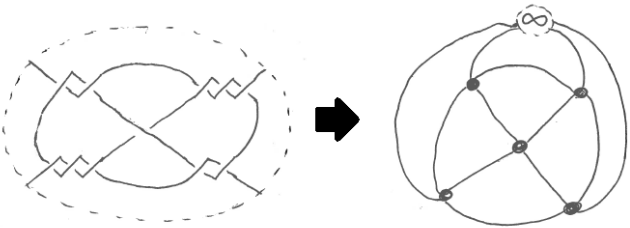
Example:



$(2/1)$	0 NW	2 NW	4 SE	5 NW	$(P:0)$
$(3/1)$	1 NE	0 NE	3 NW	4 SW	$(P:1)$
$(1/-2)$	2 SE	0 SE	5 SE	4 NW	$(P:\infty)$
$(1/-1)$	3 SW	5 NE	1SE	2 SW	$(P:1')$
$(3/1)$	1 SW	4 NE	3 SE	0 SW	$(P:1)$

Connection with the Maximal Algebraic Subtangle Graph

The algebraic planar diagram code describes the MASG.



Each generalized planar diagram code is refined from the preceding:

$$\text{PD} \rightarrow \text{IPD} \rightarrow \text{RPD} \rightarrow \text{APD}$$

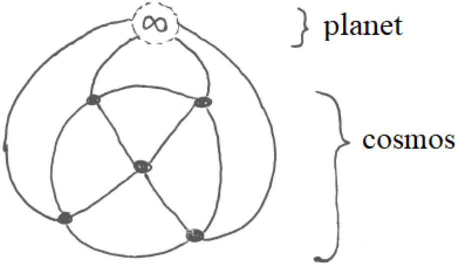
Table of Contents

A New Type of Graph: Constellations

Definition

A $k+1$ **constellation** (C,p) consists of a graph C with $k+1$ vertices which is connected, planar, 4-regular, and **simple**, and a marked $k+1^{st}$ vertex p.

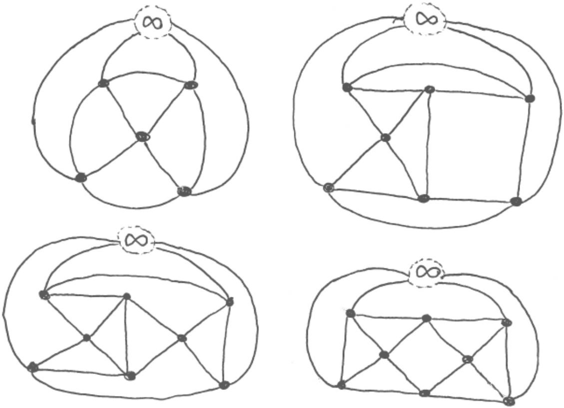
- The $k+1^{st}$ vertex p is called the **planet**.
- The induced subgraph $C^* = C - \{p\}$ is called the **cosmos**.



Constellation Examples

Definition

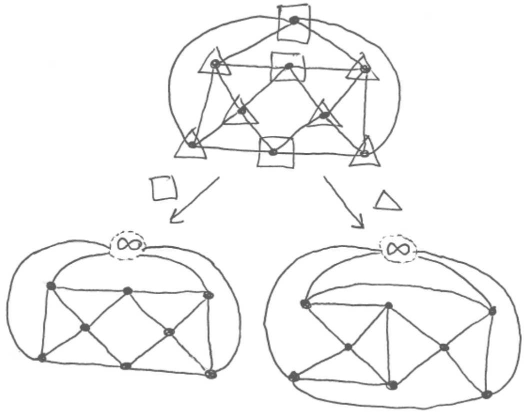
Two constellations are **equivalent** if they have isomorphic cosmos.



Changing Planets

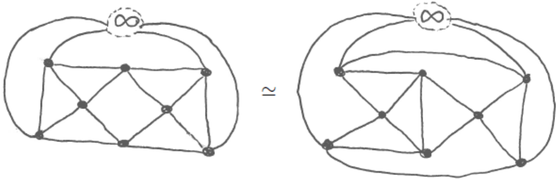
Theorem

A different choice of planet may yield non-equivalent constellations.

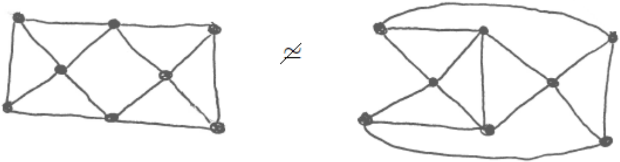


The Two 8+1 Constellations

These two graphs are isomorphic.



These two subgraphs are not.



Constellation Adjacency Matrices

Theorem

Let G and G' be graphs with adjacency matrices A and A', respectively. Then G and G' are isomorphic if and only if A and A' are permutation symmetric.

$$\begin{bmatrix} 0&1&0&1&0&0&1&0&1\\ 1&0&1&0&1&0&1&0&0\\ 0&1&0&1&1&1&0&0&0\\ 1&0&1&0&0&1&0&0&1\\ 0&1&1&0&0&0&1&1&0\\ 0&0&1&1&0&0&0&1&1\\ 1&1&0&0&1&0&0&1&0\\ 0&0&0&0&1&1&1&0&1\\ 1&0&0&1&0&1&0&1&0 \end{bmatrix} \simeq \begin{bmatrix} 0&1&0&1&0&0&1&0&1\\ 1&0&1&1&1&0&0&0&0\\ 0&1&0&0&1&1&0&1&0\\ 1&1&0&0&0&1&0&0&1\\ 0&1&1&0&0&0&1&1&0\\ 0&0&1&1&0&0&0&1&1\\ 1&0&0&0&1&0&0&1&1\\ 0&0&0&0&1&1&1&0&1\\ 1&0&0&1&0&1&1&0&0 \end{bmatrix}$$

Cosmos Adjacency Matrices

Theorem

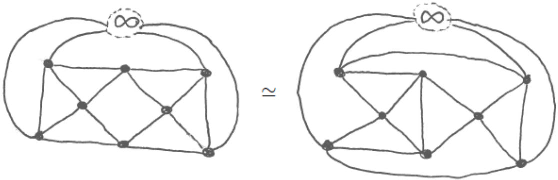
Let G be a graph with adjacency matrix A and let p be a vertex in G. If $G' = G - \{p\}$ is the subgraph of G obtained by removing p, then G' has adjacency matrix A' obtained from A by deleting the column and row corresponding to p.

$$\begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} \not\simeq \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$

The Galaxy of a Constellation

Definition

Two constellations (C_1, p_1) and (C_2, p_2) are **familiar** if their underlying graphs are isomorphic, $C_1 \cong C_2$.



Definition

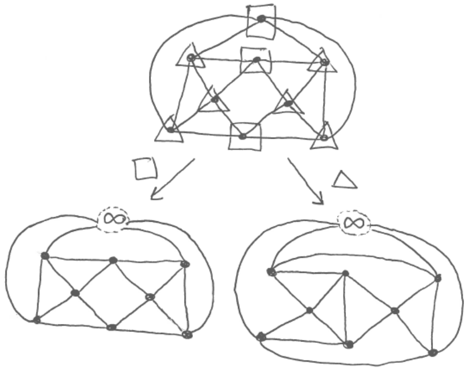
The **galaxy** of a graph[4] C is the set of equivalence classes of constellations (C, p) arising from any choice of planet p.

[4]connected, planar, 4-regular, simple

Galaxies and Familiarity

Corollary

All constellations in the same galaxy are familiar to each other.

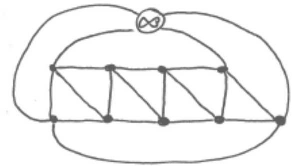


9+1 Constellations: Galaxy 1

Theorem

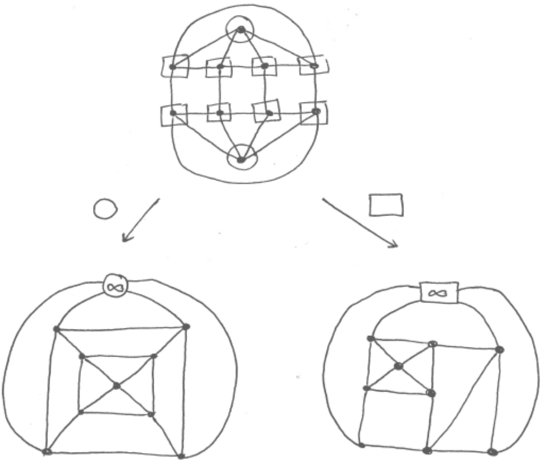
There are exactly three galaxies of 9+1 constellations.

Each 9+1 galaxy contains a different number of constellations.

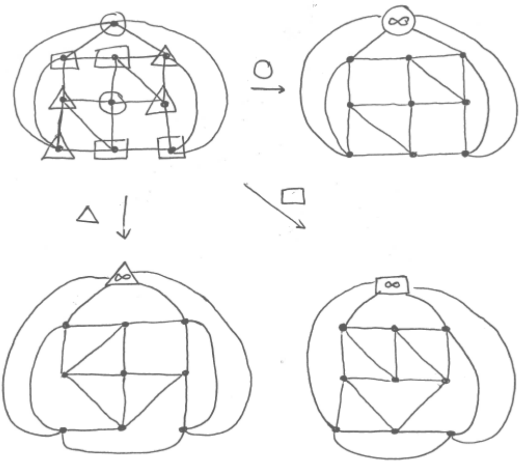


Any choice of planet in the above yields an equivalent constellation. The galaxy of this graph contains only this single constellation.

9+1 Constellations: Galaxy 2



9+1 Constellations: Galaxy 3



Combinatorial Observations

$k+1$	Number of Constellations	Number of Galaxies[5]
$5+1$	1	1
$6+1$	0	0
$7+1$	1	1
$8+1$	2	1
$9+1$	6	3
$10+1$	≥ 13	3
$11+1$	?	13
$12+1$	?	21
$13+1$	?	68
$\vdots$	$\vdots$	$\vdots$

[5] http://www.mathe2.uni-bayreuth.de/markus/reggraphs.html#PLANAR

Constellation Notation for Non-Algebraic Tangles

Definition

A non-algebraic tangle diagram can be described using sums and products of rational subtangles and constellations:

- use constellations in place of non-algebraic subtangles;
- list out the algebraic subtangle components;
- order tangles by encounter and indicate parity.

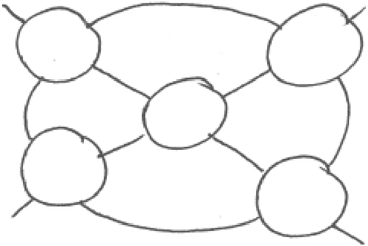
Example:

$$\underbrace{C^{(1)}_{5+1}}_{\text{constellation}} \quad \left((2/1)^0, \underbrace{(3/1)^1}_{\text{subtangle}}, (1/-2)^\infty, (1/-1)^{1'}, (3/1)^1\right)$$

The subtangles come from the algebraic planar diagram code.

Reconstructing from Notation: Cosmos

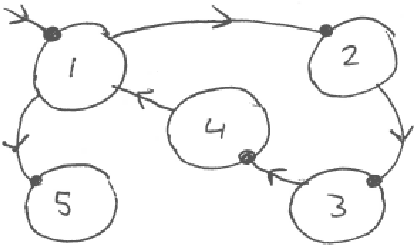
$$\underbrace{C^{(1)}_{5+1}}_{\text{constellation}} \quad \left((2/1)^0, (3/1)^1, (1/-2)^\infty, (1/-1)^{1'}, (3/1)^1\right)$$



This tangle uses the cosmos of the $5+1$ constellation.

Reconstructing from Notation: Component Order

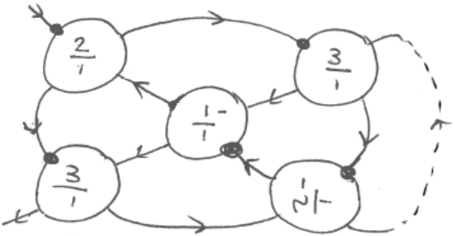
$$C^{(1)}_{5+1}\left(\underbrace{(2/1)^0}_{1},\underbrace{(3/1)^1}_{2},\underbrace{(1/-2)^\infty}_{3},\underbrace{(1/-1)^{1'}}_{4},\underbrace{(3/1)^1}_{5}\right)$$



The parity of each component forces the location of the next component. In this example, the ordered parities are $(0, 1, \infty, 1', 1)$.

Reconstructing from Notation: Filling in Components

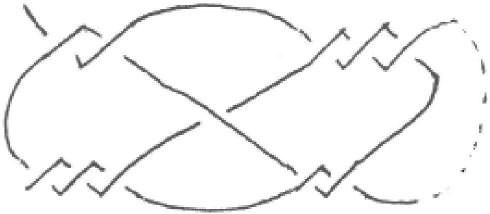
$$C_{5+1}^{(1)}\left(\underbrace{(2/1)^0}_{1},\underbrace{(3/1)^1}_{2},\underbrace{(1/-2)^{\infty}}_{3},\underbrace{(1/-1)^{1'}}_{4},\underbrace{(3/1)^1}_{5}\right)$$



Insert appropriate subtangle construction into each component.

Reconstructing from Notation: Drawing Subtangles

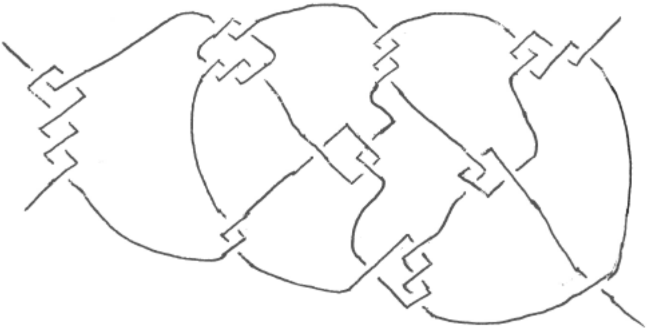
$$C_{5+1}^{(1)}\left((2/1)^0,(3/1)^1,(1/-2)^\infty,(1/-1)^{1'},(3/1)^1\right)$$



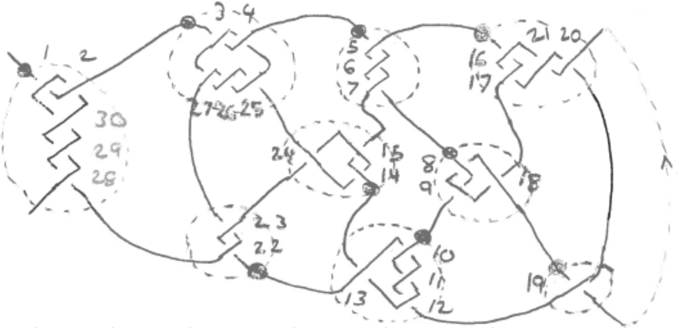
Finally, draw in algebraic subtangles given by construction.

Example: Scary Non-Algebraic Diagram

How to describe this diagram?

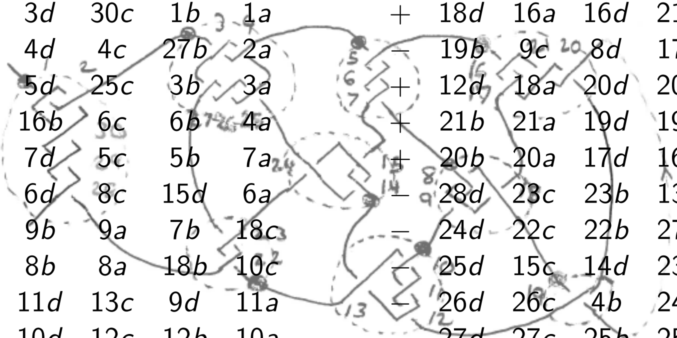


Example: Gauss Code



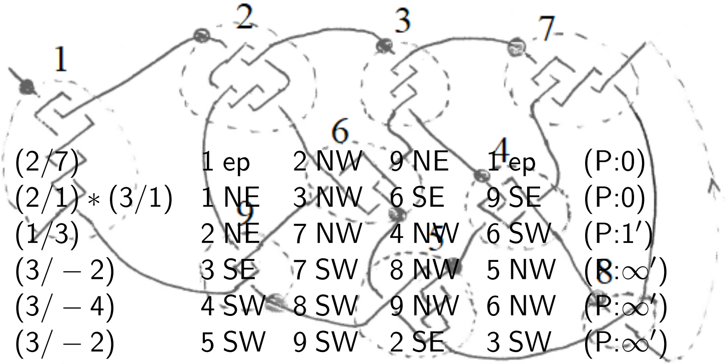
(|-b1-a2-b3-a4-b5-a6-b7+a8+b9-a10-b11-a12-b13+a14+b15-a7
-b6-a5+b16+a17-b18+a9+b8-a18+b19(|)+a20+b21+a16+b17
+a21+b20+a19-b12-a11-b10-a13-b22-a23-b24+a15+b14-a24
-b25-a26-b27-a3-b4-a25-b26-a27-b23-a22-b28-a29-b30-a1-b2-a30
-b29-a28|)

Example: Planar Diagram Code



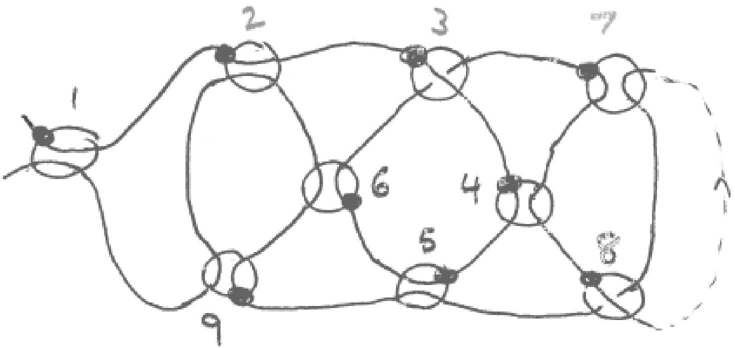
$-$	$2d$	$2c$	$30b$	$28a$	$+$	$17b$	$5a$	$21d$	$17c$
$-$	$3d$	$30c$	$1b$	$1a$	$+$	$18d$	$16a$	$16d$	$21c$
$-$	$4d$	$4c$	$27b$	$2a$	$-$	$19b$	$9c$	$8d$	$17a$
$-$	$5d$	$25c$	$3b$	$3a$	$+$	$12d$	$18a$	$20d$	$20c$
$-$	$16b$	$6c$	$6b$	$4a$	$+$	$21b$	$21a$	$19d$	$19c$
$-$	$7d$	$5c$	$5b$	$7a$	$+$	$20b$	$20a$	$17d$	$16c$
$-$	$6d$	$8c$	$15d$	$6a$	$-$	$28d$	$23c$	$23b$	$13a$
$+$	$9b$	$9a$	$7b$	$18c$	$-$	$24d$	$22c$	$22b$	$27a$
$+$	$8b$	$8a$	$18b$	$10c$	$-$	$25d$	$15c$	$14d$	$23a$
$-$	$11d$	$13c$	$9d$	$11a$	$-$	$26d$	$26c$	$4b$	$24a$
$-$	$10d$	$12c$	$12b$	$10a$	$-$	$27d$	$27c$	$25b$	$25a$
$-$	$13d$	$11c$	$11b$	$19a$	$-$	$23d$	$3c$	$26b$	$26a$
$-$	$22d$	$14c$	$10b$	$12a$	$-$	$1d$	$29c$	$29b$	$22a$
$+$	$15b$	$15a$	$13b$	$24c$	$-$	$30d$	$28c$	$28b$	$30a$
$+$	$14b$	$14a$	$24b$	$7c$	$-$	$29d$	$1c$	$2b$	$29a$

Example: Algebraic Planar Diagram Code



$(2/7)$	1 ep	2 NW	9 NE	1 ep	(P:0)
$(2/1) * (3/1)$	1 NE	3 NW	6 SE	9 SE	(P:0)
$(1/3)$	2 NE	7 NW	4 NW	6 SW	$(P{:}1')$
$(3/-2)$	3 SE	7 SW	8 NW	5 NW	$(P{:}\infty')$
$(3/-4)$	4 SW	8 SW	9 NW	6 NW	$(P{:}\infty')$
$(3/-2)$	5 SW	9 SW	2 SE	3 SW	$(P{:}\infty')$
$(5/2)$	3 NE	8 ep	8 NE	4 NE	$(P{:}\infty')$
$(1/1)$	4 SE	7 SE	7 ep	5 NE	(P:1)
$(1/2)$	5 SE	1 SE	2 SW	6 NE	$(P{:}\infty)$

Example: Algebraic Subtangle Components by Parity

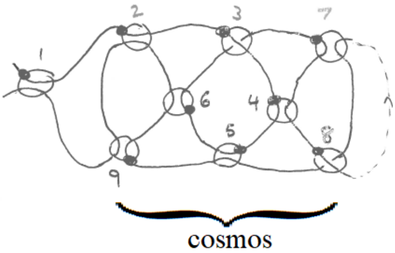


$$\text{APD Gauss Code} \quad : \quad [1,2,3,4,5,6,3,7,4,8,|7,8,5,9,6,2,9,1]$$

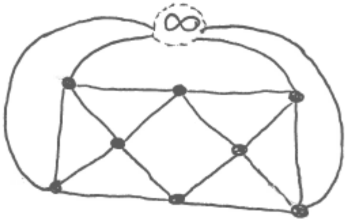
$$\text{Parity Vector} \quad : \quad (0,0,1',\infty',\infty',\infty',\infty',1,\infty)$$

Example: Identifying Cosmos of 8+1 Constellation

Subtangle Graph



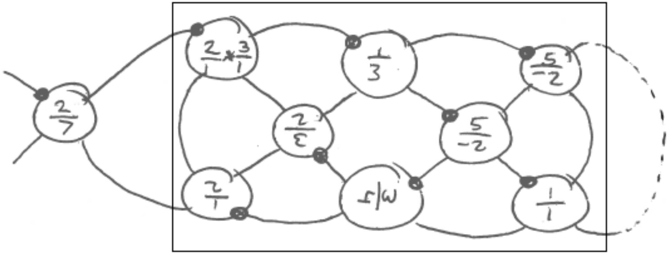
8+1 Constellation



APD Gauss Code : $[1, 2, 3, 4, 5, 6, 3, 7, 4, 8, |7, 8, 5, 9, 6, 2, 9, 1]$

Parity Vector : $(0, 0, 1', \infty', \infty', \infty', \infty', 1, \infty)$

Example: Constellation Notation



algebraic + non-algebraic

$$T = (2/7) + C^{(1)}_{8+1}\Big((2/1) * (3/1)^{0}, (1/3)^{1'}, (3/-2)^{\infty'}, (3/-4)^{\infty'},$$
$$(3/-2)^{\infty'}, (5/2)^{\infty'}, (1/1)^{1}, (1/2)^{\infty}\Big)$$

$$\text{Parity Vector}: (0, 0, 1', \infty', \infty', \infty', \infty', 1, \infty)$$

Constellation Notation Summary

Benefits:

- compact notation for non-algebraic diagrams
- combines with sum/product notation
- construction intuitive to visualize

Drawbacks:

- always assumes internally canonical components
- only helpful for known constellations
- classifies non-algebraic diagrams, **but not tangles**

Major Advantage:

preferring canonical components eliminates many redundancies